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A new mechanism for saturating unstable r modes in neutron stars

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ABSTRACT

We consider a new mechanism for damping the oscillations of a mature neutron star. The new dissipation channel arises if superfluid vortices are forced to cut through superconducting flux tubes. This mechanism is interesting because the oscillation modes need to exceed a critical amplitude in order for it to operate. Once it acts, the effect is very strong (and non-linear) leading to efficient damping. The upshot of this is that modes are unlikely to ever evolve far beyond the critical amplitude. We consider the effect of this new dissipation channel on the r modes, which may be driven unstable by the emission of gravitational waves. Our estimates show that the flux tube cutting leads to a saturation threshold for the instability that can be smaller than that of other proposed mechanisms. This suggests that the idea may be of direct astrophysical relevance.

Key words: gravitational waves – stars: neutron – stars: oscillations.

1 CONTEXT

Neutron stars represent a hands-off laboratory for physics under extreme conditions, and may ultimately provide a complement to information gleaned from particle colliders like the Large Hadron Collider. While such terrestrial experiments probe hot plasmas at relatively low densities, the core of a neutron star requires an understanding of the cold dense part of the quantum chromodynamics phase diagram (Alford et al. 2008). To gain access to this information, we need to accurately model how a realistic neutron star interior connects to its exterior and affects observable features.

A commonly considered example involves the cooling of the star. Which processes lead to the star cooling down and how does heat flow from the interior to the surface? By matching models of possible scenarios to X-ray data for isolated neutron stars, we may be able to constrain the theory. An excellent recent example of this is provided by the observed real-time cooling of the remnant in Cassiopeia A, which has provided the first true constraint on the superfluid transition temperature for the star's core (Page et al. 2011; Shternin et al. 2011).

Another aspect of the problem relates to the dynamics of the star's complex core. A neutron star undergoes a number of changes as it evolves and provided that these are dramatic enough, various stellar oscillation modes may be excited. These could, in turn, affect the emission pattern of the star (either in X-ray or radio) provided that the interior fluid motion has a significant effect on the star's magnetosphere. This has led to the development of neutron star asteroseismology, where the aim is to use future observations to probe the star's interior in the same way that helioseismologists have successfully constrained the interior physics on the Sun.

A breakthrough in this area came with the observations of quasi-periodic oscillations in the X-ray tails of large magnetar flares (Strohmayer & Watts 2005). Early, relatively naive, models suggested that the observed oscillations could be identified with various elastic oscillation modes of the star's crust (Piro 2005; Samuelsson & Andersson 2007). More recent work has attempted, not yet completely successfully, to account for the anticipated strong magnetic field effects (Colaiuda & Kokkotas 2012; Gabler et al. 2013). This is a very difficult problem, but there has been clear progress in the last few years.

Since neutron stars are distant, one would expect their oscillations to be excited to detectable amplitudes only under exceptional circumstances. Such events would be rare, like the magnetar flares. However, there is an exception to this rule. Modes of oscillation may become unstable at various instances during the star's life. Provided an unstable mode is allowed to grow large enough, such instabilities may lead to a detectable signal and may also have an indirect effect on the star's evolution (say of the spin). A number of possible instabilities have been discussed in the literature. As far as mature neutron stars are concerned, the most promising ideas involve the Coriolis-driven r modes, which somewhat counter-intuitively may become unstable due to the gravitational waves they emit (Andersson 1998; Friedman & Morsink 1998). This has stimulated

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a large body of work on the nature of the r modes, the gravitational wave signal they would be associated with and the physics that may affect the growth of the instability. A number of possible damping and saturation mechanisms have been suggested over the last 15 years or so (Bildsten & Ushomirsky 2000; Rezzolla, Lamb & Shapiro 2000; Arras et al. 2003; Glampedakis & Andersson 2006; Nayyar & Owen 2006; Bondarescu, Teukolsky & Wasserman 2007, 2009; Haskell, Andersson & Passamonti 2009; Andersson, Haskell & Comer 2010; Haskell & Andersson 2010; Alford, Mahmoodifar & Schwenzer 2012; Gusakov, Chugunov & Kantor 2014). Nevertheless, the conclusions from state-of-the-art modelling remain relatively unaffected. The r-mode instability is likely to set a spin-threshold for neutron stars. This is an important observation since the fastest observed radio pulsars and accreting neutron stars spin well below the theoretical break-up limit (Chakrabarty et al. 2003; Patruno 2010). A mechanism is required to explain this, and the r-mode instability appears to fit the bill. Furthermore, a recent analysis of the problem has shown that the theoretical predictions for the r-mode instability window for a 'minimal' neutron star model, which does not include superfluity or the appearance of exotic particles (such as hyperons or deconfined quarks) in the core, is not consistent with current X-ray observations of low-mass X-ray binaries (LMXBs; Ho, Andersson & Haskell 2011; Haskell, Degenaar & Ho 2012). There is, therefore, a clear need to include additional effects in our modelling, such as superfluity and superconductivity in the core of the star.

This paper introduces a new mechanism to the r-mode scenario. The argument involves the star's core and builds on the fact that there is likely to be a region where superfluid neutron vortices coexist with superconducting protons. As has been argued in different contexts, such a region may have decisive impact on the star's dynamics. Due to the interaction between superfluid vortices and magnetic flux tubes, any changes in the star's vorticity (the bulk rotation or the fluid motion associated with an oscillation mode) may be coupled to the magnetic field. This suggests two scenarios. In the first, the vortices become pinned to the, more plentiful, flux tubes. In the second scenario, the vortices can cut through the flux tubes, but at a cost. This latter process is expected to be highly dissipative. It is this possibility that we explore in this paper.

2 BRIEF SUMMARY

The fact that superfluid dynamics is damped by a mutual friction arising from the interaction between quantized vortices and other components in the mixture (typically, phonons in laboratory studies of He_4 and electrons in a neutron star core) is well established. The main idea dates back to work by Hall and Vinen (Hall & Vinen 1956). They introduced a linear friction between superfluid (helium) vortices and the 'normal' component (represented by phonons). Balancing this force by the Magnus force that would drive the vortices to move along with the superfluid condensate in the absence of friction, they deduced the functional form for the force, $\boldsymbol{F}_{\rm mf}$ in the following, that couples the two 'fluid' components in the system: the superfluid condensate and the normal component.

In the standard picture, the vortex friction, $\boldsymbol{f}_{\rm D}$, is taken to be *linear* in the relative velocity $\boldsymbol{u}$ between the vortices and the normal fluid:

$$\boldsymbol{f}_{\rm D} = \rho_{\rm n}\kappa\mathcal{R}\boldsymbol{u}, \qquad (1)$$

where $\rho_{\rm n}$ and κ are, respectively, the density of the superfluid and the quantum of circulation associated with each vortex. The dimensionless friction coefficient, $\mathcal{R}$, is assumed to be velocity independent. The force balance that controls the motion of individual vortices leads to a linear algebraic relation $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{w})$, where $\boldsymbol{w}$ is the relative velocity between the condensate and the normal fluid. Inverting this relation, one finds that the relative fluid flow is damped according to

$$\partial_t \boldsymbol{w} + \{\cdots\} = -\frac{1}{x_{\rm p}\rho_{\rm n}}\boldsymbol{F}_{\rm mf}, \qquad (2)$$

where $\boldsymbol{F}_{\rm mf}$ is obtained from $\boldsymbol{f}_{\rm D}$ by using the inferred $\boldsymbol{u}(\boldsymbol{w})$ relation and combining the effect for an array of vortices. The brackets in equation (2) represent fluid terms that are not relevant to this discussion and $x_{\rm p} = \rho_{\rm p}/\rho$ is the normal fluid fraction ($\rho = \rho_{\rm p} + \rho_{\rm n}$ is the total density).

In the case of superfluid neutron star dynamics, one can show that a similar relation applies provided that $\rho_{\rm n}$ and $\rho_{\rm p}$ are taken to be the neutron and proton densities. Hence, a relation like equation (2) will affect the relative motion associated with any global oscillation mode. This means that we can extract a characteristic mutual friction dissipation time-scale in terms of the mode energy $E_{\rm mode}$ (obtained as a volume integral of the inviscid velocity field) and the rate of work $\dot{E}_{\rm mf}$ done by $\boldsymbol{F}_{\rm mf}$. Provided the damping rate is slow compared to the dynamics of the mode, the time-scale is well approximated by

$$\tau_{\rm mf} = \frac{2E_{\rm mode}}{|\dot{E}_{\rm mf}|}. \qquad (3)$$

This time-scale can be very short if the mode under consideration has a significant counter-moving component. Detailed work has shown that this is the case for the fundamental f mode, and as a result the gravitational-wave-driven instability of this mode is severely suppressed in a superfluid star (Lindblom & Mendell 1995; Andersson, Glampedakis & Haskell 2009). The conclusion for the Coriolis-restored r mode is different. The r modes are affected by mutual friction to a much lesser extent, essentially because they are mainly horizontal (Lee & Yoshida 2003; Haskell, Andersson & Passamonti 2009; Passamonti, Haskell & Andersson 2009).

The mechanism we consider in this paper is subtly different from the Hall–Vinen model in that the friction force turns out to be *non-linear* in the relative flow $\boldsymbol{u}$. This means that the inferred mode damping, still expressed in terms of an $\mathcal{R}$ coefficient, will be velocity dependent. Whenever this is the case, the problem has interesting new aspects. Most importantly, the equation for the relative motion (equation 2) becomes non-linear, which means that the mutual friction may be able to prevent a given oscillation mode from growing beyond some threshold amplitude. That is, in addition to damping the mode, the mutual friction may lead to the saturation of an instability.

In the following section, we outline the derivation of the new friction force. The steps involved essentially repeat the analysis of Link (2003). Having done this, we will discuss the implications for the r-mode instability. Readers that are mainly interested in the astrophysical results (or may already be familiar with the flux tube cutting mechanism) can proceed straight to Section 4.

3 THE NEW FRICTION MECHANISM

3.1 Vortex–flux tube pinning

The interaction between superfluid neutron vortices and superconducting proton flux tubes in the outer core of a neutron star is thought to be key to the evolution of the system, possibly linking changes in spin to the evolution of the magnetic field. An important ingredient in this problem is the energy cost associated with superfluid vortices, which are expected to be magnetized due to the entrainment effect (Alpar, Langer & Sauls 1988), cutting through superconducting flux tubes. As a rough estimate, one may consider the energy associated with superposition of a neutron vortex and a proton flux tube. This leads to what we will refer to as the pinning energy, $f_{\rm pin}$, acting on each moving vortex. Ignoring geometrical factors related to direction dependence, the force per intersection is of the order of (Ruderman, Zhu & Chen 1998)

$$F_{\rm int} \approx \frac{E_{\rm int}}{\Lambda_*} = \Lambda_*^2 B_{\rm n} B_{\rm p}, \tag{4}$$

where the London penetration length Λ_* (which is of the order of few tens of fm) represents the typical size of the overlap region, while $B_{\rm n}$ and $B_{\rm p}$ are the magnetic fields carried by individual vortices and flux tubes, respectively. The force per unit length of a given vortex is then

$$f_{\rm pin} \approx \frac{F_{\rm int}}{d_{\rm p}}, \tag{5}$$

where

$$d_{\rm p} \approx \left(\frac{B}{\phi_0}\right)^{1/2} \approx 3 \times 10^3 B_{12}^{-1/2}\ {\rm fm}. \tag{6}$$

Here, B (and $B_{12} = B/10^{12}$ G) is the macroscopic core magnetic field, ϕ_0 is the quantum of magnetic flux and $d_{\rm p}$ is the typical distance separating the flux tubes.

This estimate allows us to quantify how easy it is for a vortex to cut through the array of flux tubes in a neutron star core. A necessary condition is that vortices do not pin to the flux tubes, which means that the Magnus force must exceed the pinning force. To make this quantitative, let us represent the vortex and flux tube velocities by $\boldsymbol{u}_{\rm n}$ and $\boldsymbol{u}_{\rm p}$, respectively. Meanwhile, the macroscopic flows (that enter the averaged two-fluid hydrodynamics) are given by $\boldsymbol{v}_{\rm n}$ (for the superfluid neutrons) and $\boldsymbol{v}_{\rm p}$ (for the proton condensate). If a vortex is pinned to the flux tubes, then we expect to have $\boldsymbol{u}_{\rm n} = \boldsymbol{u}_{\rm p} \approx \boldsymbol{v}_{\rm p}$. Basically, it is natural to assume that the flux tubes move with the proton condensate. This means that the velocity difference that enters into the Magnus force is approximated by $\boldsymbol{u}_{\rm n} - \boldsymbol{v}_{\rm n} \approx \boldsymbol{v}_{\rm p} - \boldsymbol{v}_{\rm n} \equiv \boldsymbol{w}$. Given this, we can obtain a *minimum* velocity lag, $w_{\rm pin}$, between the neutron and proton fluids *below which* vortex pinning is likely to take place (Link 2003):

$$w_{\rm pin} \approx \frac{f_{\rm pin}}{\rho_{\rm n}\kappa} \approx 1.5 \times 10^4\ B_{12}^{1/2}\ {\rm cm\,s^{-1}}. \tag{7}$$

We note here that in this expression (and the ones hereafter) only the dependence with respect to the magnetic field is shown while the fluid density has been set to $\rho = 10^{14}$ g cm^{-3}, a value representative of a neutron star outer core.

The estimate (7) will be of central importance later. The key point is that, as long as the relative velocity w between the two fluids is below $w_{\rm pin}$, the vortices will not be able to move relative to the flux tubes. Hence, the damping mechanism that we will now discuss will not act.

3.2 Kelvin-wave damping

Once the pinning can no longer balance the Magnus force and the vortices start moving, they must cut through the flux tube array to keep going. This may be a highly dissipative process due to the excitation of Kelvin waves along the vortex. This point was first argued by Epstein & Baym (1992) for vortices moving through the lattice of nuclei in the star's crust, and later adapted by Link (2003) to the conditions in the core that we discuss here. In an effective theory, the waves on the vortex can be treated as particles, 'kelvons', with effective mass μ and energy $E_{\rm k} = \hbar^2 k^2/2\mu$, where k is the associated wavenumber. If we let

$$\boldsymbol{u} = \boldsymbol{u}_{\rm p} - \boldsymbol{u}_{\rm n} \tag{8}$$

be the relative vortex–flux tube velocity, then the interaction at each intersection lasts a time interval $t_{\rm int} \sim \Lambda_*/u$. The kelvon energy can be estimated by using this characteristic time-scale in the standard formula for an oscillator; $E_{\rm k} \approx \hbar/t_{\rm int}$ (ultimately originating from the uncertainty principle). This then leads to the characteristic wavenumber

$$k \approx \left(\frac{2\mu}{\hbar\Lambda_*} u\right)^{1/2} \equiv \frac{1}{\Lambda_*}\left(\frac{u}{v_\Lambda}\right)^{1/2}. \tag{9}$$

Given that the characteristic velocity is

$$v_\Lambda = \hbar/2\mu\Lambda_* \approx 10^9\ {\rm cm\,s^{-1}}, \tag{10}$$

we should typically have $k\Lambda_* \ll 1$ in the case of neutron star dynamics. A more sophisticated analysis, leading to the same final estimate, can be found in Link (2003).

In order to calculate a dissipation rate, we need the kelvons produced at different intersections of the same vortex to add incoherently. This requires $kd_{\rm p} \gg 1$, which in turn leads to a lower limit for the relative vortex–flux tube velocity:

$$u_{\rm low} \approx 6.5 \times 10^5\ B_{12}\ {\rm cm\,s^{-1}}. \tag{11}$$

In order for the mechanism we discuss to operate efficiently, we need $u \gg u_{\rm low}$. Note that $u_{\rm low} > w_{\rm pin}$ when $B \gtrsim 10^8$ G or so. The estimates we present are thus still consistent for the case of LMXBs, as long as the *internal* magnetic field is not much stronger than the inferred *exterior* dipolar magnetic field strength (which is inferred to be $\approx 10^8$ G). If we want to consider significantly stronger magnetic fields, we would need to first understand the behaviour at velocities in the range between $w_{\rm pin}$ and $u_{\rm low}$ better.

The energy released at each vortex/flux tube intersection was determined by Link (2003). The result is

$$\Delta E = \frac{2}{\pi}\frac{F_{\rm int}^2}{\rho_{\rm n}\kappa}(v_\Lambda u)^{-1/2}. \tag{12}$$

This suggests that the energy loss rate (per unit volume) is

$$\dot{\mathcal{E}}_{\rm cut} = \frac{\mathcal{N}_{\rm n} u}{d_{\rm p}^2}\Delta E, \tag{13}$$

where $\mathcal{N}_{\rm n}$ is the number of vortices per unit area. Alternatively, we can use the fact that (ignoring entrainment, which only affects the estimate by a factor of order unity; Andersson, Sidery & Comer 2006) $2\Omega_{\rm n} \approx \mathcal{N}_{\rm n}\kappa$, to get

$$\dot{\mathcal{E}}_{\rm cut} = \frac{4\Omega_{\rm n}}{\pi\rho_{\rm n}\kappa^2} f_{\rm pin}^2 \left(\frac{u}{v_\Lambda}\right)^{1/2}. \tag{14}$$

We can relate the rate (equation 14) to the work done by a drag force (exerted on a unit length vortex segment) of the general form

$$\boldsymbol{f}_{\rm D} = \rho_{\rm n}\kappa\mathcal{R}\boldsymbol{u} \tag{15}$$

with a velocity-dependent coefficient $\mathcal{R} = \mathcal{R}(u)$. From this, we can construct a hydrodynamical mutual friction force density exerted on the neutron fluid by averaging over the vortex array,

$$\boldsymbol{F}_{\rm mf} = \mathcal{N}_{\rm n}\boldsymbol{f}_{\rm D} \quad \rightarrow \quad \boldsymbol{F}_{\rm mf} = 2\Omega_{\rm n}\rho_{\rm n}\mathcal{R}(u)\boldsymbol{u}. \tag{16}$$

Then, from

$$\dot{\mathcal{E}}_{\rm cut} = \boldsymbol{F}_{\rm mf} \cdot \boldsymbol{u}, \tag{17}$$

we infer that

$$\mathcal{R} = \mathcal{R}_0 \left(\frac{v_\Lambda}{u}\right)^{3/2} \tag{18}$$

with

$$\mathcal{R}_0 = \frac{2}{\pi}\left(\frac{f_{\rm pin}}{\rho_{\rm n}\kappa v_\Lambda}\right)^2 \quad \rightarrow \quad \mathcal{R}_0 \approx 1.4 \times 10^{-10}\, B_{12}. \tag{19}$$

The key observation here is that, as soon as the vortices start to move relative to the flux tubes they are likely to be prevented by a very strong friction. This is obvious since $w_{\rm pin}$ and $u_{\rm low}$ are both going to be much smaller than v_Λ. The lower the relative velocity, the stronger this damping is. In practice, this means that the vortices are unlikely to be able to keep moving and the system will be driven back towards pinning.

4 r-MODE DAMPING AND SATURATION

In the previous section, we outlined the argument that leads to vortices cutting though flux tubes being a highly dissipative process. This argument is not original, but we believe this is the first time that the discussion has been framed in the context of a mutual friction force. The final result (equation 16) allows us to consider the mechanism in a range of relevant contexts. For example, once the dissipation due to vortex–flux tube cutting is expressed as a mutual friction force, we can adapt it for the two-fluid hydrodynamics model used to model neutron star oscillations and instabilities. As an illustration of this analysis, let us try to estimate what the effect on the gravitational-wave-driven r-mode instability may be.

In order to make use of the deduced mutual friction force in a problem involving the standard two-fluid model, we first of all need to replace the dependence on the relative velocity, $\boldsymbol{u}$, between vortices and flux tubes with the relative velocity, $\boldsymbol{w}$, between the two macroscopic fluid components. The standard approach to this, pioneered by Hall and Vinen more than half a century ago (Hall & Vinen 1956), is to first balance the vortex force (equation 16) by the Magnus force that acts on the vortices and invert the relation to get an expression for $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{w})$. The steps involved are straightforward in the case where the friction coefficient $\mathcal{R}$ is constant. When $\mathcal{R}$ is velocity dependent, the analysis becomes slightly more involved and one should in principle consider the full problem, including relative flows in the background. However, in the present case, we can bypass this problem by making a couple of (potentially debatable) assumptions.

First of all, on dynamical time-scales, the flux tubes can be assumed to move with the protons, which means that $\boldsymbol{u}_{\rm p} \approx \boldsymbol{v}_{\rm p}$. It is not quite so easy to justify a similar relation between the neutron fluid and vortex velocities. To make progress, we nevertheless *assume* that $\boldsymbol{u}_{\rm n} \approx \boldsymbol{v}_{\rm n}$. This would be true for free vortices and it might be a reasonable approximation in the case of vortices moving at high speed through the flux tube array. This is, in fact, the approximation underpinning the model in Section 3 so it make sense to make this approximation here as well. With these assumptions, we simply have $\boldsymbol{u} = \boldsymbol{w}$.

Now, from detailed two-fluid calculations (Haskell et al. 2009), we know that unstable r modes have a particular relative velocity contribution. In general, this contribution is position dependent, due to the density dependence of the superfluid pairing gaps. In order to keep things simple, we will nevertheless assume that this contribution is proportional to the average velocity perturbation, $\boldsymbol{v}$. This leads to

$$w = \lambda v \quad \rightarrow \quad w \approx \lambda\alpha\left(\frac{r}{R}\right)^2 \Omega R, \tag{20}$$

where α is the usual (dimensionless) r-mode amplitude (e.g. Owen et al. 1998). We know from actual mode calculations that the counter-moving contribution enters at higher order in the slow-rotation expansion such that

$$\lambda = \lambda_0 \left(\frac{\Omega}{\Omega_{\rm K}}\right)^2, \tag{21}$$

where $\Omega_{\rm K}$ is the break-up frequency and λ_0 is taken to be a spin-independent factor.

If the mode has large enough amplitude to force vortices through flux tubes, then $w \gtrsim w_{\rm pin}$ which means that

$$\mathcal{R} \lesssim 2.5 \times 10^{-3}\, B_{12}^{1/4}. \tag{22}$$

It is worth noting that the deduced upper limit, depending on the magnetic field strength, is about a factor of ∼10–100 larger than the drag coefficient associated with the standard mutual friction mechanism: scattering of electrons by vortices (Alpar et al. 1988; Andersson et al. 2006). Note that a direct comparison to such a mechanism can, however, be misleading. In our case, the drag coefficient is velocity dependent (thus introducing a different velocity dependence in the damping integrals) and, furthermore, we will be calculating the r-mode damping time-scale using the eigenfunctions and counter-moving velocity amplitudes obtained in the strong pinning limit.

The damping time-scale can be estimated in the usual way (see e.g. Andersson & Kokkotas 2001) by making use of equation (3). This argument involves the r-mode energy

$$E_{\rm mode} \approx \frac{1}{2}\alpha^2\Omega^2 M R^2 \tilde{J}, \qquad \text{where} \qquad \tilde{J} = \frac{1}{MR^4}\int_0^R \rho r^6 {\rm d}r, \tag{23}$$

which leads to $\tilde{J} = 0.016$ for an $n = 1$ polytrope (Owen et al. 1998). The mutual friction damping rate is given by

$$\dot{E}_{\rm mf} = \int \dot{\mathcal{E}}_{\rm cut} {\rm d}V \tag{24}$$

and our estimates lead to

$$\dot{E}_{\rm mf} \approx \frac{4\Omega}{\pi\kappa^2}\int \frac{f_{\rm pin}^2}{\rho}\left(\frac{w}{v_\Lambda}\right)^{1/2} {\rm d}V. \tag{25}$$

That is,

$$\dot{E}_{\rm mf} \approx \frac{8\Omega^{5/2}}{\pi\nu_{\rm K}}\frac{f_{\rm pin}^2}{\kappa^2}\left(\frac{\alpha\lambda_0}{Rv_\Lambda}\right)^{1/2}\int_{R_{\rm in}}^R \frac{r^3}{\rho}{\rm d}r, \tag{26}$$

where $\nu_{\rm K} = \Omega_{\rm K}/2\pi$. We have assumed that flux tube cutting takes place in the outer part of the stellar core, in the region $R_{\rm in} < r \lesssim R$ (where the coexistence of a neutron superfluid and a proton superconductor is likely) and that λ_0 and ρ are approximately uniform.

Through these arguments, we obtain an order of magnitude estimate for the mutual friction damping time-scale (using $\nu_{\rm K} \approx 1233$ Hz for canonical stellar parameters $M = 1.4\,{\rm M}_\odot$, $R = 10^6$ cm) as

$$\tau_{\rm mf} \approx 6 \times 10^{10} \lambda_0^{-1/2}\alpha^{3/2}\nu_{500}^{-1/2}B_8^{-1}\ {\rm s}, \tag{27}$$

where $\nu_{500} = \nu/500$ Hz is the scaled spin frequency of the star ($\nu = \Omega/2\pi$).

If we want to consider the relevance of the proposed mechanism for various astrophysical scenarios, then we need to provide an estimate for λ_0. This will require a more detailed numerical calculation for realistic superfluid parameters, etc. However, we can use previous mode calculations to get an idea of the likely range of values for this parameter. Extracting an averaged value from the r-mode study by Haskell et al. (2009, assuming their pinning limit), we find that λ_0 ought to lie in the range

$$\langle\lambda_0\rangle \approx 0.1{-}1 \tag{28}$$

both for strong and weak superfluidity models. This result is obviously not very precise, but it will allow us to assess whether the new damping mechanism is strong enough to warrant a more detailed investigation.

In considering possible astrophysical scenarios, it is important to appreciate that the features of the new mechanism are rather different from the standard mutual friction. Most importantly, the dissipation due to flux tube cutting is a non-linear process that *saturates* but does not completely suppress an unstable mode. This mutual friction mechanism does not operate as soon as w is driven down to $w_{\rm pin}$ when vortices can repin to the flux tubes. Hence, one would expect an unstable mode to evolve in such a way that its amplitude saturates around this level.[1] This provides a rough estimate of the r-mode amplitude of such systems:

$$w \approx w_{\rm pin} \quad \rightarrow \quad \alpha_{\rm pin} \approx 10^{-6}\left(\frac{\lambda_0}{0.1}\right)^{-1} \nu_{500}^{-3} B_8^{1/2}. \tag{29}$$

We can use this threshold amplitude to rewrite the mutual friction time-scale (equation 27) in a more transparent form:

$$\tau_{\rm mf} \approx 190\left(\frac{\lambda_0}{0.1}\right)^{-2}\left(\frac{\alpha}{\alpha_{\rm pin}}\right)^{3/2} \nu_{500}^{-5} B_8^{-1/4}\ {\rm s}. \tag{30}$$

This time-scale is (at least) about an order of magnitude shorter than the mode's growth time-scale, assuming an $n = 1$ polytropic star (Andersson & Kokkotas 2001). It is therefore likely that the scenario outlined above works: once the mode amplitude exceeds $\alpha_{\rm pin}$, the unpinned vortex array is driven through the flux tubes and the ensuing friction quickly damps out the mode, effectively suppressing it back to $\alpha_{\rm pin}$.

5 ASTROPHYSICS: APPLICATION TO ACCRETING SYSTEMS

An obvious astrophysical setting where the flux tube cutting scenario may apply is in fast spinning accreting neutron stars in LMXBs. From previous considerations of the r-mode instability in this context (Brown & Ushomirsky 2000), we know that the mode amplitude required to achieve torque balance is

$$\alpha_{\rm acc} \approx 1.3\times10^{-7}\left(\frac{L_{\rm acc}}{10^{35}\ {\rm erg\ s^{-1}}}\right)^{1/2} \nu_{500}^{-7/2}. \tag{31}$$

[1] In reality, the damping mechanism will only operate during a fraction of the oscillation, corresponding to an instantaneous amplitude $|w(t)| > w_{\rm pin}$. In principle, this effect would result in a weakened dissipation and a lower time-averaged damping rate. However, the analysis of a one-dimensional toy model consisting of a clamped vibrating string suggests that, unless the oscillation amplitude is very close to the dissipation cut-off amplitude, the time-averaged damping rate is not seriously affected by this effect. We can therefore ignore it in our analysis.

Balancing the two mechanisms, as would be appropriate if the flux tube cutting allows the r mode to grow to the precise amplitude required to prevent further spin-up in an accreting system, we have

$$\frac{\alpha_{\rm pin}}{\alpha_{\rm acc}} \approx 8\left(\frac{\lambda_0}{0.1}\right)^{-1} B_8^{1/2}\left(\frac{L_{\rm acc}}{10^{35}\ {\rm erg\ s^{-1}}}\right)^{-1/2} \nu_{500}^{1/2}. \tag{32}$$

In order for the new mechanism to play a role in explaining the observed population, one would expect to have $\alpha_{\rm pin}/\alpha_{\rm acc} \approx 1$ for the fastest spinning systems.

As an example, let us consider 4U 1608+522 which spins at 620 Hz and for which the averaged accretion luminosity is 5×10^{36} erg s^{-1}. If the range we have suggested for λ_0 is reliable, then we find that the proposed scenario would work provided the *interior* magnetic field in this system is

$$B \approx (0.9 - 3)\times 10^8\ {\rm G}. \tag{33}$$

This is in the range of the expected *surface* fields for these mature systems. Moreover, it is natural to assume that the interior field (which may initially be much stronger than the externally visible field) of an old neutron star would be of the same order of magnitude as that in the exterior. The main point here is that our rough estimates lead to a result that appears consistent with both observations and our understanding of these systems. This makes it plausible that the new mechanism does, indeed, have a role to play in the r-mode scenario. At the very least, it warrants a more detailed investigation.

It is also worth noting an alternative strategy. We could take λ_0 as a 'free parameter', which would make sense given our general ignorance of the conditions in the outer core of a neutron star. This parameter could then be constrained by observations relating to the magnetic field of fast spinning accreting neutron stars. As an example of this strategy, let us consider the data for IGR J00291+5934 (taking the observational constraints from Patruno 2010). In this case, we have a spin frequency of 600 Hz, a luminosity of 6×10^{36} erg s^{-1} and a suggested external field of $B \approx 2 \times 10^8$ G from the spin-down rate in quiescence. From equation (32), we find that the accretion torque could be balanced by the flux tube cutting mechanism as long as

$$\lambda_0 \approx 0.16 \tag{34}$$

comfortably inside the range suggested by the mode calculations. Again, this example suggests that the new mechanism should be relevant.

Finally, it is interesting to compare the mode amplitude $\alpha_{\rm pin}$ (essentially the saturation amplitude associated with flux tube cutting) against previous results on r-mode saturation due to non-linear couplings with other inertial modes. In general, different saturation mechanisms would be competing with each other, and the one leading to the smallest mode amplitude would be physically the most relevant.

The first incarnation of r-mode saturation by non-linear mode-couplings is the model of Arras et al. (2003); that work predicts a maximum r-mode amplitude as

$$\alpha_{\rm A} \approx 1.4\times10^{-3}\nu_{500}^{5/2}. \tag{35}$$

This result has been refined by the more recent calculations of Bondarescu et al. (2007, 2009), resulting in a saturation amplitude $\alpha_{\rm sat} \approx 0.1\alpha_{\rm A}$ [for simplicity, we retain the spin dependence of equation (35) but we note that the behaviour of the Bondarescu et al. (2007, 2009) saturation amplitude shows a rather complicated behaviour as a function of time].

Comparing this more recent mode-coupling saturation amplitude with our $\alpha_{\rm pin}$, we obtain

$$\frac{\alpha_{\rm pin}}{\alpha_{\rm sat}} \approx 10^{-2}\left(\frac{\lambda_0}{0.1}\right)^{-1} B_8^{1/2}\nu_{500}^{-11/2}. \tag{36}$$

This suggests that the flux tube cutting mechanism is competitive as an r-mode saturation mechanism, being at least as efficient as mode-coupling. This result supports our earlier claim that a more detailed investigation of the physics of flux tube–vortex interaction as a source of friction for the r-mode instability is needed.

6 CONCLUDING DISCUSSION

To summarize our results, we have formulated a new type of vortex mutual friction force, based on the dissipative cutting of flux tubes by fast moving vortices, and have studied its impact on the r-mode instability in superfluid neutron stars. The non-linear dependence of this force with respect to the relative vortex–flux tube velocity leads to a rapid damping of the r mode above a threshold amplitude at which the vortex array is forced to unpin from the flux tubes. Effectively, this flux tube cutting friction provides a natural saturation mechanism for the r-mode instability.

We have highlighted the fact that our results may have important implications for the physics of accreting neutron stars in LMXBs. We have shown that the saturation amplitude due to flux tube cutting (represented by $\alpha_{\rm pin}$, see equation 29) could be smaller than the maximum amplitude set by non-linear couplings between the r mode and other inertial modes. Remarkably, this same amplitude could also be comparable to that required for balancing the accretion spin-up torque. In practice, this means that the saturation amplitude we calculate is such that it may allow for gravitational wave emission to be setting the spin equilibrium period for some systems (thus making them potential candidates for gravitational wave detection). However, our amplitude may also be small enough to never allow the mode to grow to the point where gravitational wave emission would influence the spin evolution (or, indeed, thermal evolution) of the system. This would allow a system to 'live' inside the standard r-mode instability window, without the need of additional damping mechanisms to explain the observations of Haskell et al. (2012) and Mahmoodifar & Strohmayer (2013). Our qualitative analysis thus shows that the saturation amplitude due to flux tube cutting is clearly in a very interesting range that can have diverse astrophysical consequences.

A more sophisticated treatment of the problem is of course necessary to accurately predict the relative strength of gravitational wave, accretion and electromagnetic spin-down torques. This is of key importance for gravitational wave detection, given that recent analysis have shown that the dynamics of many sources is probably dictated by electromagnetic and accretion torques, with only a few systems likely to be interesting targets for next-generation gravitational wave detectors (Haskell & Patruno 2011; Patruno, Haskell & D'Angelo 2012; Mahmoodifar & Strohmayer 2013).

There are aspects of the flux tube-cutting friction that have not been discussed in any detail here. For instance, an important issue is the fact that, as any other frictional force, the mechanism discussed here should provide an additional source of heating in the stellar interior. However, calculating the rate of heating is a difficult task because the quasi-stationary state of the system is likely to be that of pinning. This 'pinning regime' may not actually translate to physically immobilized vortices. The system's finite temperature may drive vortex creep with $u \sim w_{\rm pin}$. Unfortunately, in this velocity regime, our analysis breaks down, making it impossible to make any prediction about dissipation and heating. We can, nevertheless, obtain an upper limit for the heating rate by using the mode damping rate of the cutting regime. By then balancing the energy dissipation rate in equation (14) with the energy carried away by neutrino emission due to Cooper paring, $\dot{E}_{\rm cp} = 1.5 \times 10^{31} T_8^8 \,{\rm erg\,s^{-1}}$, with T_8 the temperature in units of 10^8 K, we obtain core temperatures of the order of 10^8 K. This temperature is consistent with the observed surface temperatures of LMXBs, especially for the faster systems which are also likely to be the most interesting for gravitational wave emission (Haskell et al. 2012; Mahmoodifar & Strohmayer 2013).

A more detailed understanding of vortex–flux tube interactions over the entire range of the expected velocities would represent a key advance in this area, relevant for many aspects of neutron star dynamics. The mechanism we have discussed may not only be crucial for our understanding of the non-linear development of the r-mode instability, it could also impact on models of pulsar glitches (Link 2012; Haskell, Pizzochero & Seveso 2013) and the combined magnetorotational evolution of neutron stars (Ruderman et al. 1998; Glampedakis & Andersson 2011; Glampedakis, Andersson & Samuelsson 2011). To make further progress, we need to sharpen our computational tools and develop models that account for the mesoscopic vortex–flux tube interactions while, at the same time, track the macroscopic fluid dynamics. This is a challenging problem but the estimates we have presented provide clear motivation for future efforts.

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