## Chapter 17. The Kerr solution
### 17.1. Remarks on the derivation

The Kerr metric emerged during an investigation of algebraically special vacuum solutions of Einstein's equations, with the intention to find a physically meaningful generalisation of the Schwarzschild solution [187,188].

Then, it was found to describe the exterior gravitational field of a rotating body.

No exact solution of Einstein's equations with a perfect fluid source is known that could be matched to the Kerr metric. Its main application is to rotating black holes.

No invariant criterion leading uniquely to it is known.

[187] R. P. Kerr and A. Schild, in *Atti del Convegno sulla Relativit`a Generale: Problemi dell'Energia e Onde Gravitazionali*. G. Barbera Editore, Firenze 1965, pp. 1. Reprinted in *Gen. Relativ. Gravit.* **41**, 2485 (2009), with an editorial note by A. Krasiński, E. Verdaguer and R. P. Kerr, *Gen. Relativ. Gravit.* **41**, 2469 (2009) and authors' biographies by A. Krasiński, *Gen. Relativ. Gravit.* **41**, 2482 (2009) (Kerr); and L. Shepley, *Gen. Relativ. Gravit.* **8**, 955 (1977).

[188] R. P. Kerr and A. Schild, in *Golden Oldies in general relativity. Hidden gems*. Edited by A. Krasiński, G. F. R. Ellis, and M. A. H. MacCallum, Springer, Heidelberg 2013, p. 439.

Deriving the Kerr metric from the Einstein equations would require too much space as for a short course. See Refs. [187-189] for derivations by two different methods; both methods are presented in Ref. [11].

The Kerr solution was first announced in 1963 [190].

It is the simplest exact solution of Einstein's equations that describes the exterior field of a rotating body.

Consequently, it became a basis for hundreds of papers discussing astrophysical aspects of black holes.

Also, it is believed to be the unique asymptotic state toward which all nonstationary uncharged black holes should evolve.

[189] B. Carter, in: *Black holes – les astres occlus*. Edited by C. de Witt and B. S. de Witt. Gordon and Breach, New York, London, Paris 1973, p. 61. Reprinted in *Gen. Relativ. Gravit.* **41**, 2874 (2009), with an editorial note by N. Kamran and A. Krasiński, *Gen. Relativ. Gravit.* **41**, 2867 (2009) and author's (auto)biography by B. Carter, *Gen. Relativ. Gravit.* **41**, 2870 (2009).
[11] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006.
[190] R. P. Kerr, *Phys. Rev. Lett.* **11**, 237 (1963).

## 17. 2. Basic properties

The most compact representation of the Kerr metric is in the Boyer – Lindquist [191] (BL) coordinates:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \qquad (17.12) \end{aligned}$$

The symbols are:

$\Sigma = r^2 + a^2 \cos^2 \vartheta,\qquad \Delta_r = r^2 - 2mr + a^2.$

By comparison with the weak-field limit, the constant $a$ was identified as the angular momentum per unit mass of the body that generates the field (17.12).

The limit $a = 0$ of this is seen to be the Schwarzschild solution, so m is the mass of the source. The limit m = 0 is the Minkowski metric, and then $\varphi$ becomes the azumuthal coordinate. For the interpretation of r and $\vartheta$ see below.

[191] R. H. Boyer and R. W. Lindquist, *J. Math. Phys.* **8**, 265 (1967).

$$\begin{aligned} \mathrm{d}s^2 &= \left(1-\frac{2mr}{\Sigma}\right)\mathrm{d}t^2+\frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi-\left(\frac{2mra^2\sin^2\vartheta}{\Sigma}+r^2+a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2-\Sigma\mathrm{d}\vartheta^2. \end{aligned} \tag{17.12}$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta_r = r^2 - 2mr + a^2$.

In these coordinates the metric is independent of $x^0 = t$ (so it is *stationary*) and of $x^3 = \varphi$, so $k^\alpha{}_{(t)} = \delta^\alpha{}_3$ and $k^\alpha{}_{(\varphi)} = \delta^\alpha{}_3$ are its Killing vector fields.

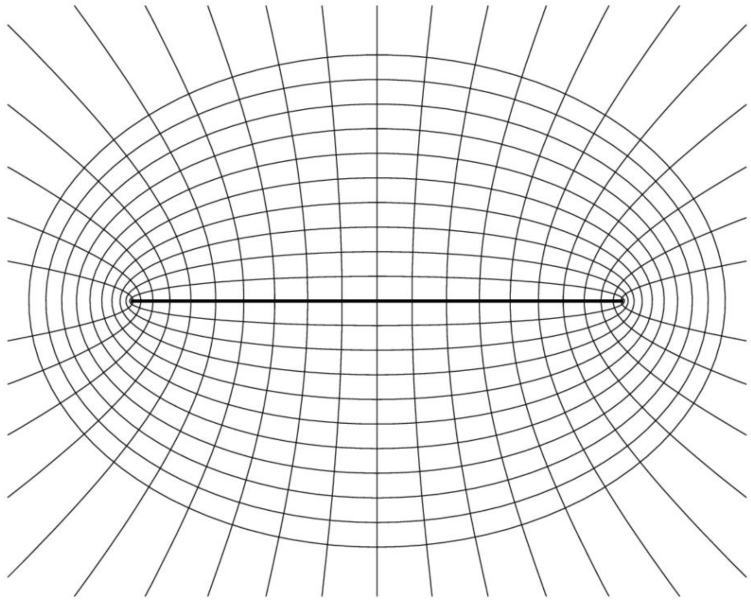
The Abelian group generated by $k^\alpha{}_{(t)} = \delta^\alpha{}_3$ and $k^\alpha{}_{(\varphi)} = \delta^\alpha{}_3$ is the complete symmetry group of (17.12), as can be verified by solving the Killing equations.

In the Minkowski limit m = 0, $\varphi$ becomes the azimuthal angle, so the Kerr metric is axially symmetric.

In the same limit, the surfaces of constant t and r become confocal ellipsoids of revolution, with their foci lying on the ring ($r = 0$, $\vartheta = \pi/2$), see next page.

The surfaces of constant t and $\vartheta$ become one-sheeted hyperboloids of revolution with the foci on the same ring (next page).

This ring is a singularity of the coordinates of (17.12), and, as we shall see below, a singularity of the spacetime as well.



An axial cross-section through a space t = constant in the Minkowski spacetime in the coordinates of (17.12).

The ellipses are cross-sections through the surfaces of constant r.

The hyperbolae are cross-sections through the surfaces of constant $\vartheta$, they are everywhere orthogonal to the ellipses.

The ellipsoids and the hyperboloids all have their foci on the ring of radius $|a|$, seen here as the thick horizontal line; it has the equation ($r = 0$, $\vartheta = \pi/2$).

The ($r$, $\vartheta$, $\varphi$) coordinates are singular on this ring. When $\vartheta = \pi/2$, the hyperboloid degenerates to the $z = 0$ plane with the disc $r = 0$ removed.

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \end{aligned} \tag{17.12}$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta_r = r^2 - 2mr + a^2$.

Since the BL coordinates ($r$, $\vartheta$, $\varphi$) are orthogonal in each hypersurface $t$ = constant, the surfaces $\varphi$ = constant are, in the Minkowski limit, planes orthogonal to the ellipsoids and to the hyperboloids.

The Schwarzschild metric has a spurious singularity at $r = 2m$, where $g_{rr} \to \infty$ and $g_{tt} = 0$.

In the Kerr metric, the set where $g_{rr} \to \infty$ (i. e., $\Delta_r = 0$) is different from $g_{tt} = 0$.

The first one exists only when $a^2 \leq m^2$. When $a^2 \neq m^2$, it consists of two disjoint parts,

$$r = r_\pm \stackrel{\text{def}}{=} m \pm \sqrt{m^2 - a^2} \tag{17.13}$$

As we will see later, it is also a spurious singularity.

It is seen from (17.12) that when $a^2 < m^2$, $r$ becomes time in those regions where $\Delta_r < 0$.

→ Just as in the Schwarzschild solution, it is impossible to stay at constant $r$ there.

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \end{aligned} \tag{17.12}$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta_r = r^2 - 2mr + a^2$.

Unlike in the Schwarzschild metric, when $a^2 < m^2$ there exists a region where $\Delta_r > 0$ but $g_{tt} \leq 0$.

In it, timelike vectors must have a nonzero $\varphi$-component whose minimal value is found from $g_{\mu\nu}v^\mu v^\nu = 0$ with $v^r = v^\vartheta = 0$ using (17.12):

$$v^\varphi_{\min} = v^0 \frac{2mra - \varepsilon\Sigma\sqrt{\Delta_r}/\sin\vartheta}{2mra^2\sin^2\vartheta + \Sigma\left(r^2 + a^2\right)}$$

where $v^0$ is the t-component and $\varepsilon = a/|a|$.

As expected, $v_{\min}^\varphi \to 0$ when $g_{tt} \to 0$, i.e. when $\Sigma \to 2mr$ and $\Delta_r \to a^2 \sin^2 \vartheta$.

The other solution of $g_{\mu\nu}v^\mu v^\nu = 0$ at $v^r = v^\vartheta = 0$, with + in the numerator, corresponds to the opposite generator of the same light cone and does not go to 0 when $g_{tt} \to 0$.

The presence of a $\varphi$-component of a timelike geodesic vector is a signature of ***frame dragging*** in the gravitational field of rotating bodies.

This effect does not exist in the Newtonian gravitation: the exterior gravitational field of a rotating body ``feels'' the rotation and forces the orbiting bodies to drift in the azimuthal direction.

It was first calculated, perturbatively but in a general metric, by Hans Thirring and Josef Lense in 1918 [192].

An experiment to measure it (using a gyroscope orbiting the Earth) was proposed by Leonard Schiff in 1960 [193].

The experiment, under the name of ***Gravity Probe B*** and under the direction of C. W. Francis Everitt, was carried out in the years 2004 – 2005 [194] after preparations that lasted more than 40 years [195]; see a semi-popular account in Ref. [196].
(BTW, the result confirmed the prediction of general relativity.)

[192] H. Thirring, *Phys. Zeitschr.* **19**, 33 (1918); **22**, 29 (1921); J. Lense and H. Thirring, *Phys. Zeitschr.* **19**, 156 (1918). Reprinted in B. Mashhoon, F.W. Hehl and D.S. Theiss, Gen. Relativ. Gravit. 16, 711 (1984) and in Ref. [188].
[193] L. Schiff, *Phys. Rev. Lett*. **4**, 215 (1960); *Proc. Nat. Acad. Sci. USA* **46**, 871 (1960).
[194] C. W. F. Everitt et al., *Phys. Rev. Lett*. **106**, 221101 (2011).
[195] C. W. F. Everitt, in *Experimental Gravitation. Proceedings of the International School of Physics ``Enrico Fermi'', course 56.* Edited by B Bertotti, Academic Press, New York 1974, p. 331.
[196] https://en.wikipedia.org/wiki/Gravity Probe B

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \end{aligned} \tag{17.12}$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta_r = r^2 - 2mr + a^2$.

The t-coordinate in (17.12) coincides with the proper time of an observer at infinity.

The ratio of ds of the local observer at rest in the set $g_{tt} = 0$ to the corresponding ds at infinity is zero → the light emitted from $g_{tt} = 0$ arrives to a distant observer with $z = \infty$.

The set $g_{tt} = 0$ is sometimes called *infinite redshift hypersurface*.

But in the Kerr metric, this *is not* the surface of a black hole, as we will see later.

However, as pointed out by Carter [189], this name is misleading.

An observer at rest in the B-L coordinates has r, $\vartheta$ and $\varphi$ all constant.

On the hypersurface $g_{tt} = 0$ and inside it (where $g_{tt} < 0$), for such an ``observer'' $ds^2 \le 0$, → he/she would have to move with the velocity of light or faster just to remain at rest relative to infinity.

→ Stationary observers do not exist where $g_{tt} \le 0$.

A more appropriate name for the set $g_{tt} = 0$ is ***stationary limit hypersurface***.

$\Sigma = r^2 + a^2 \cos^2 \vartheta, \qquad \Delta_r = r^2 - 2mr + a^2.$

For the calculation below we assume that $\Delta_r \geq 0$.

Let us project the Riemann tensor on the orthonormal basis defined by [189]

$$\begin{aligned} e^0{}_\alpha \mathrm{d}x^\alpha &= \sqrt{\frac{\Delta_r(r)}{\Sigma(r)}} \left(\mathrm{d}t - a\sin^2\vartheta \mathrm{d}\varphi\right), \qquad e^1{}_\alpha \mathrm{d}x^\alpha = \sqrt{\frac{\Sigma(r)}{\Delta_r(r)}}\mathrm{d}r, \\ e^2{}_\alpha \mathrm{d}x^\alpha &= \sqrt{\Sigma(r)}\mathrm{d}\vartheta, \qquad e^3{}_\alpha \mathrm{d}x^\alpha = \sin\vartheta\, \frac{a\mathrm{d}t - (r^2 + a^2)\,\mathrm{d}\varphi}{\sqrt{\Sigma(r)}}. \end{aligned} \tag{17.14}$$

The components of these projections are

$$\begin{aligned} R_{0101} &= -R_{2323} = 2I_1, \\ R_{0123} &= 2R_{0213} = -2R_{0312} = -2I_2, \\ R_{0202} &= R_{0303} = -R_{1212} = -R_{1313} = -I_1, \\ I_1 &\overset{\text{def}}{=} mr\, \frac{r^2 - 3a^2\cos^2\vartheta}{\Sigma^3}, \\ I_2 &\overset{\text{def}}{=} ma\cos\vartheta\, \frac{3r^2 - a^2\cos^2\vartheta}{\Sigma^3}. \end{aligned} \tag{17.15}$$

All these components are finite where $\Delta_r = 0$.

The singularity is located on the ring $\{r = 0, \vartheta = \pi/2\}$, where $\Sigma = 0$.

But the interior of the ring is nonsingular, so the metric can be extended through this set, to negative values of r.

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 \\ &- \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \end{aligned} \tag{17.12}$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta, \qquad \Delta_r = r^2 - 2mr + a^2.$

When $\vartheta$ is near to $\pi/2$, while $r < 0$ but sufficiently near to zero, then $2mra^2 \sin^2 \vartheta/\Sigma$ in $g_{\varphi\varphi}$ of (17.12) becomes negative and large in absolute value.

Then $\varphi$ becomes a timelike coordinate and the curves of constant t, $\vartheta$ and r in that region are timelike.

If we require that, by continuity, these lines are closed with the period $2\pi$, then it follows that ***closed timelike curves exist in the $r < 0$ sheet of the extended Kerr manifold*** [191, 189].

This applies to all three varieties of the Kerr solution, $|a| < m$, $|a| = m$ and $|a| > m$; see further on.

$$r = r_{\pm} \stackrel{\mathrm{def}}{=} m \pm \sqrt{m^2 - a^2} \qquad (17.13)$$

## 17. 3. The event horizons and the stationary limit hypersurfaces

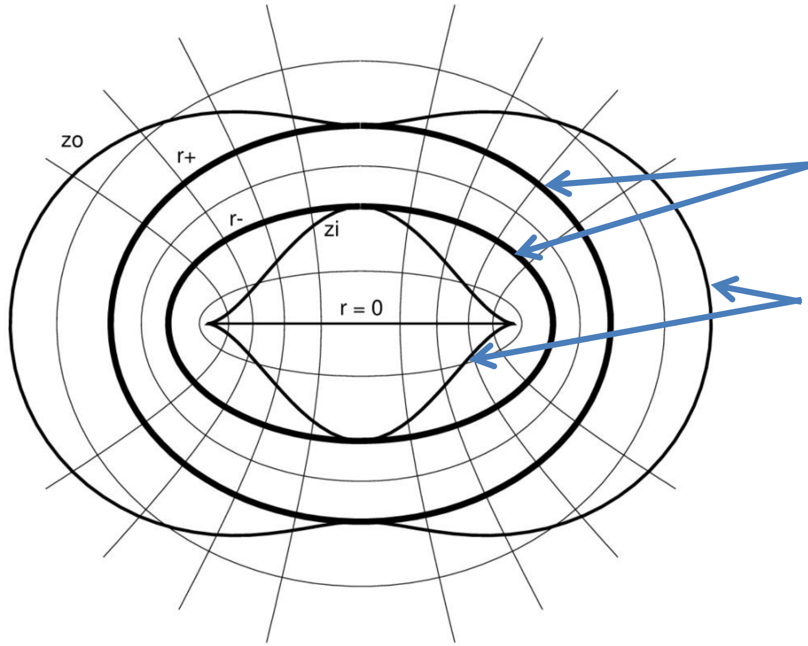
We have already noted that the hypersurfaces given by (17.13), if they exist, play a special role. As we will see later, they are ***event horizons*** (EHs) .

We also noted that the stationary limit hypersurfaces (SLHs) given by

$$r^2 - 2mr + a^2 \cos^2 \vartheta = 0 \Longrightarrow r = m \pm \sqrt{m^2 - a^2 \cos^2 \vartheta} \qquad (17.16)$$

play another special role.

We will now consider the shapes of these two families of surfaces and their relation to each other.




Event horizons:

$$r = r_{\pm} \stackrel{\text{def}}{=} m \pm \sqrt{m^2 - a^2}$$

Stationary limit hypersurfaces:

$$r = z_{o,i} \stackrel{\text{def}}{=} m \pm \sqrt{m^2 - a^2 \cos^2 \vartheta}$$

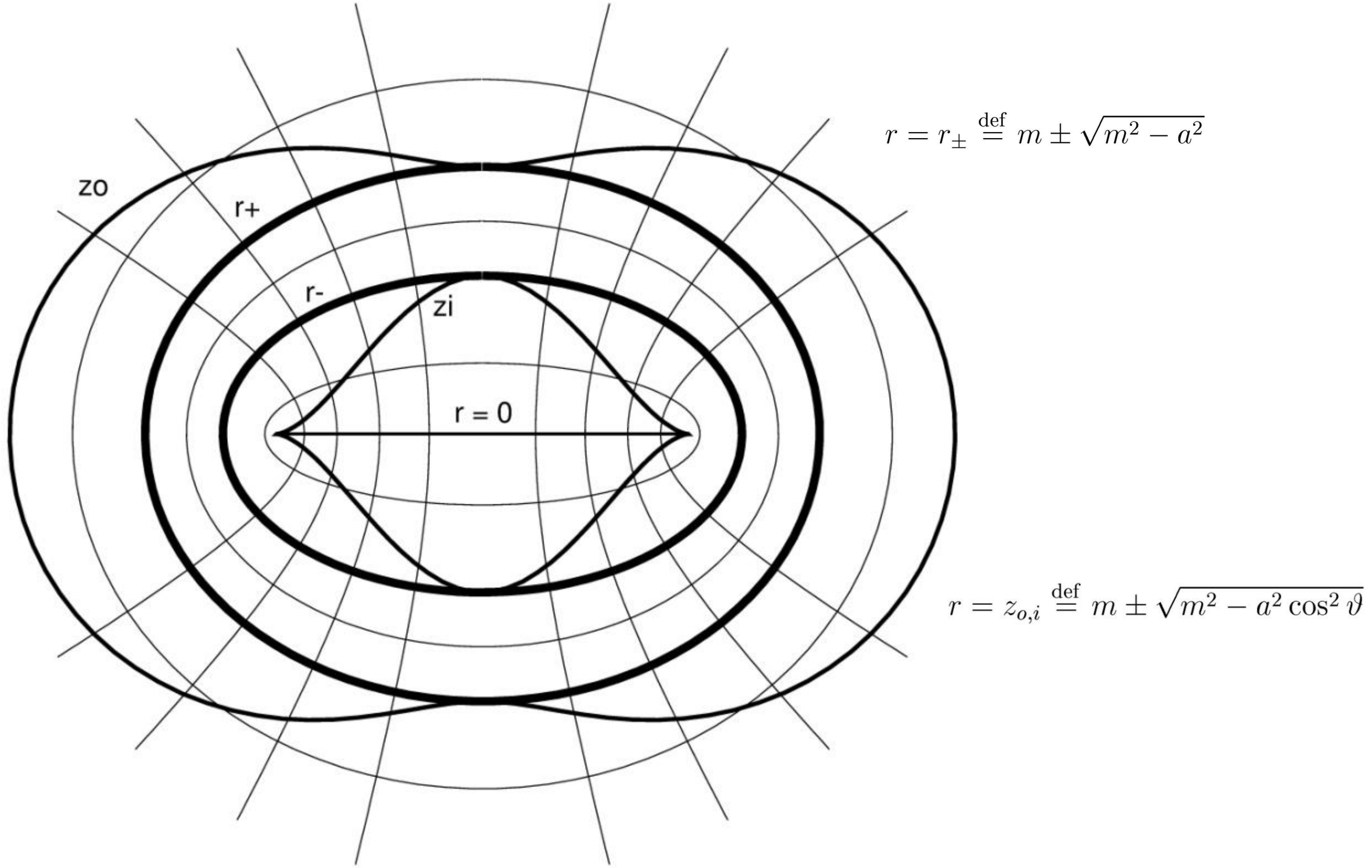
When $a^2 < m^2$, there are two event horizons; the one at $r = r_-$ is inside that at $r = r_+$.

There are also two stationary limit hypersurfaces.

The outer one, at $r = z_o = m + \sqrt{m^2 - a^2 \cos^2 \vartheta}$ envelops the outer event horizon, and is tangent to it only at the axis, where $\vartheta = 0$ or $\vartheta = \pi$.

The inner stationary limit surface, at $r = m - \sqrt{m^2 - a^2 \cos^2 \vartheta}$, lies within the inner event horizon and is tangent to it also only at the axis.

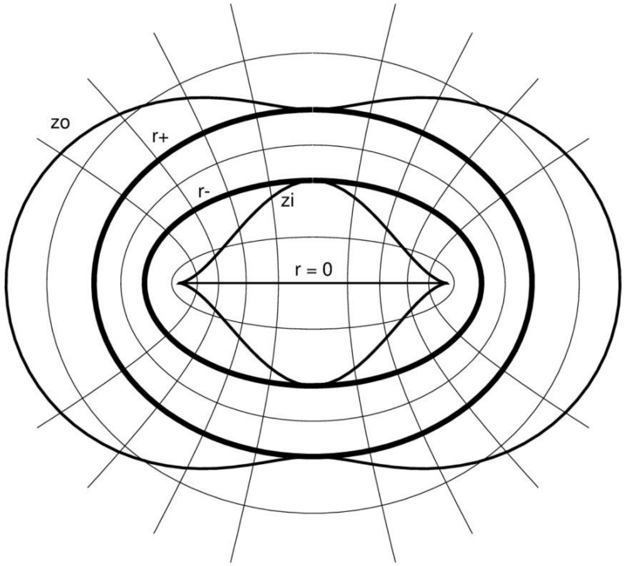
It is tangent to the disc $r = 0$ at its singular edge, where it has its own edge.




As $a \to 0$, the Kerr solution tends to the Schwarzschild solution.

Then, the inner stationary limit surface and the inner event horizon both shrink to a point, together with the disc r = 0.

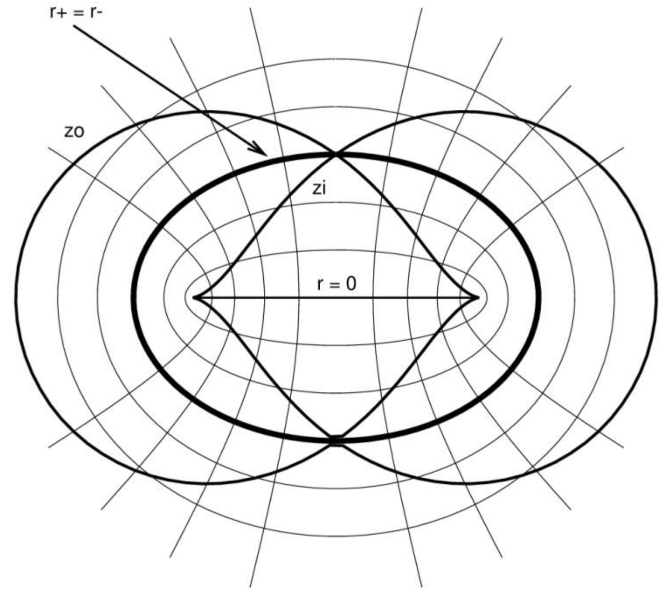
The outer stationary limit surface and the outer event horizon coalesce and go over into the Schwarzschild horizon at r = 2m.




$$r = r_{\pm} \stackrel{\text{def}}{=} m \pm \sqrt{m^2 - a^2}$$

$|a| \rightarrow m$

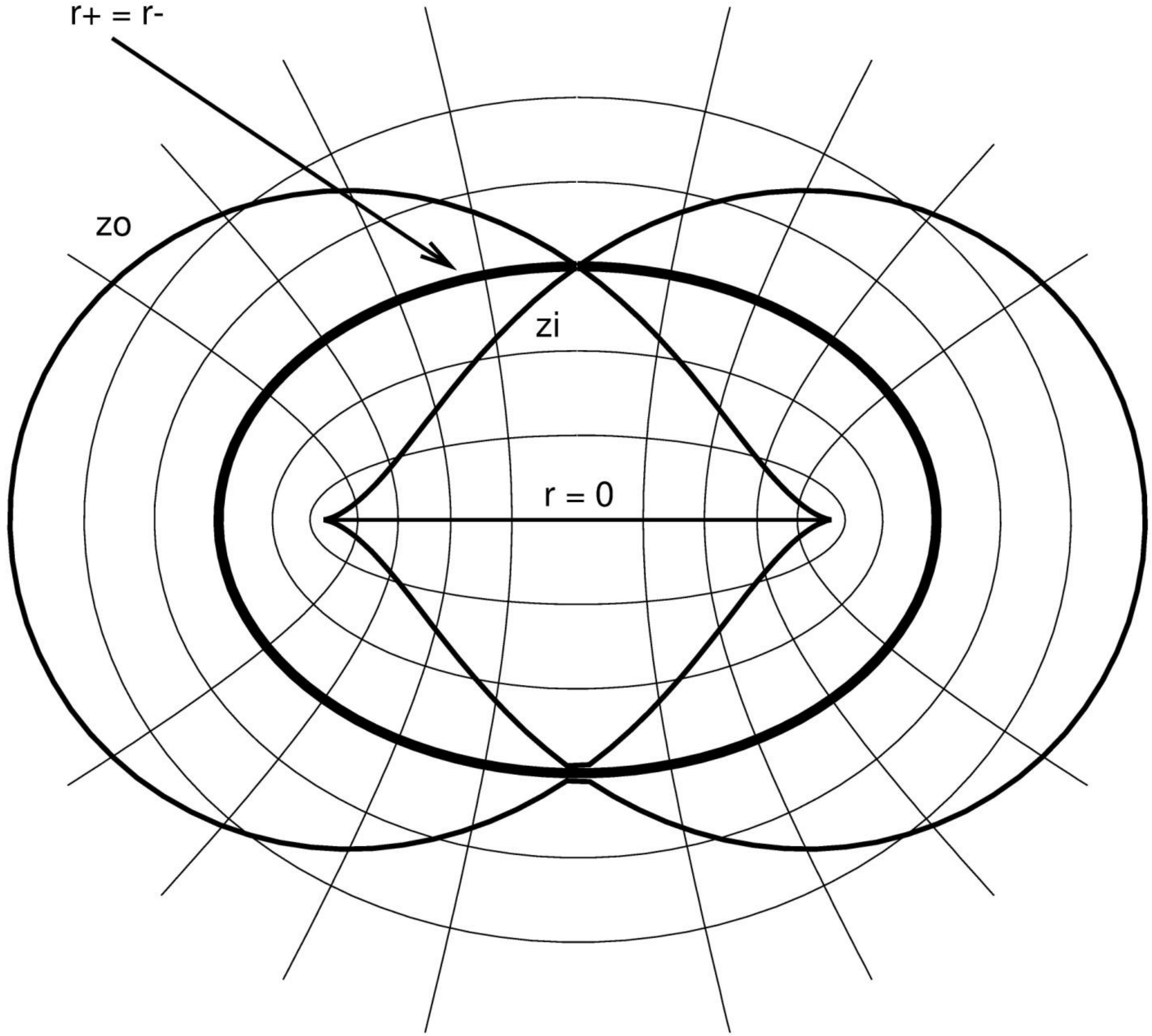
$$r = z_{o,i} \stackrel{\text{def}}{=} m \pm \sqrt{m^2 - a^2 \cos^2 \vartheta}$$




As $|a| \rightarrow$ m, the event horizons approach each other to meet at r = m when $a^2 = m^2$.

The concave regions of the outer stationary limit surface shrink to points, and the surface becomes conical in their neighbourhoods, the vertices of the cones touching the event horizon.

Similarly, the inner stationary limit surface becomes conical in the neighbourhood of the axis of symmetry, the vertices of the cones being common with the outer surface.


r+ = r-
zo
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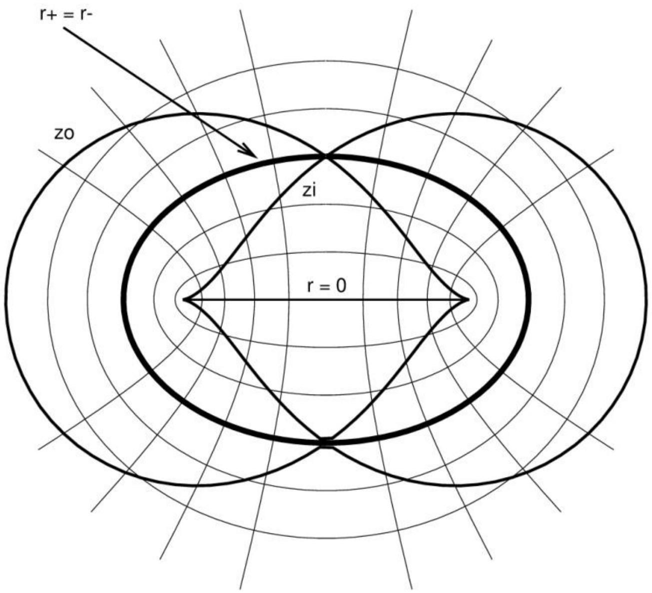
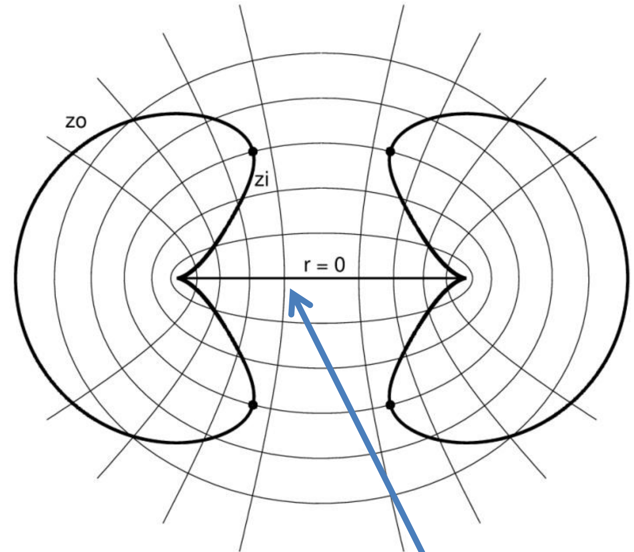
r = 0

$a^2 = m^2$




$a^2 > m^2$




When $a^2 > m^2$, the event horizon disappears completely.

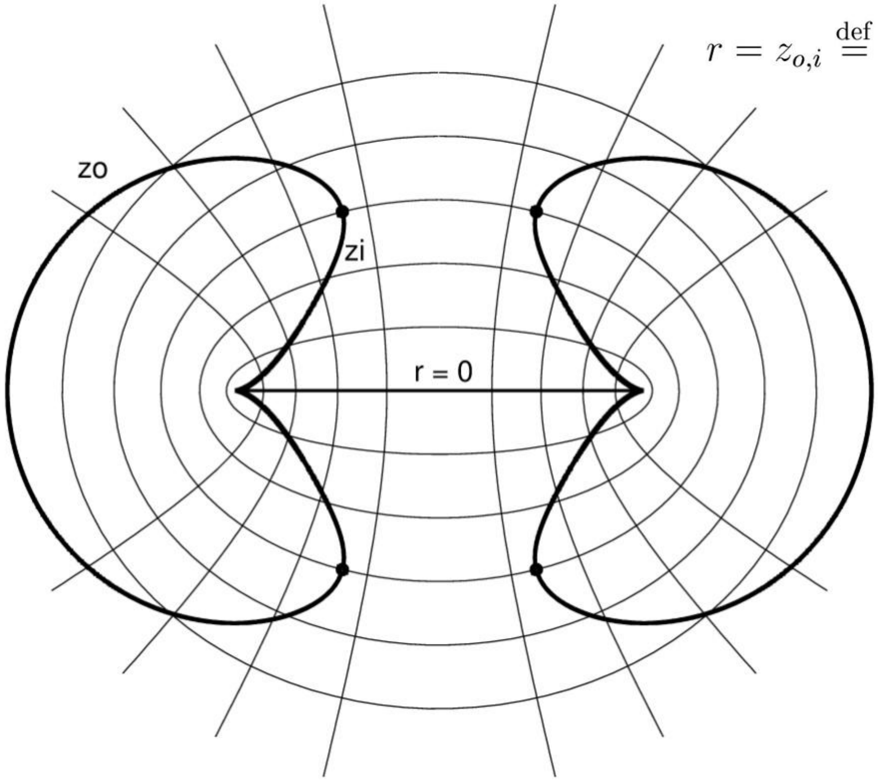
→The singular ring at {r = 0, $\vartheta = \pi/2$} is accessible and becomes a ***naked singularity***.

This case, similarly to the previous one, has no Schwarzschild limit.

The conical points of the stationary limit surfaces change to open holes, and the two stationary limit surfaces become parts of one surface that has the topology of a torus.

The hole in the surface around the axis of symmetry is the larger, the greater the difference $a^2 - m^2$.

$$r = z_{o,i} \stackrel{\mathrm{def}}{=} m \pm \sqrt{m^2 - a^2 \cos^2 \vartheta}$$



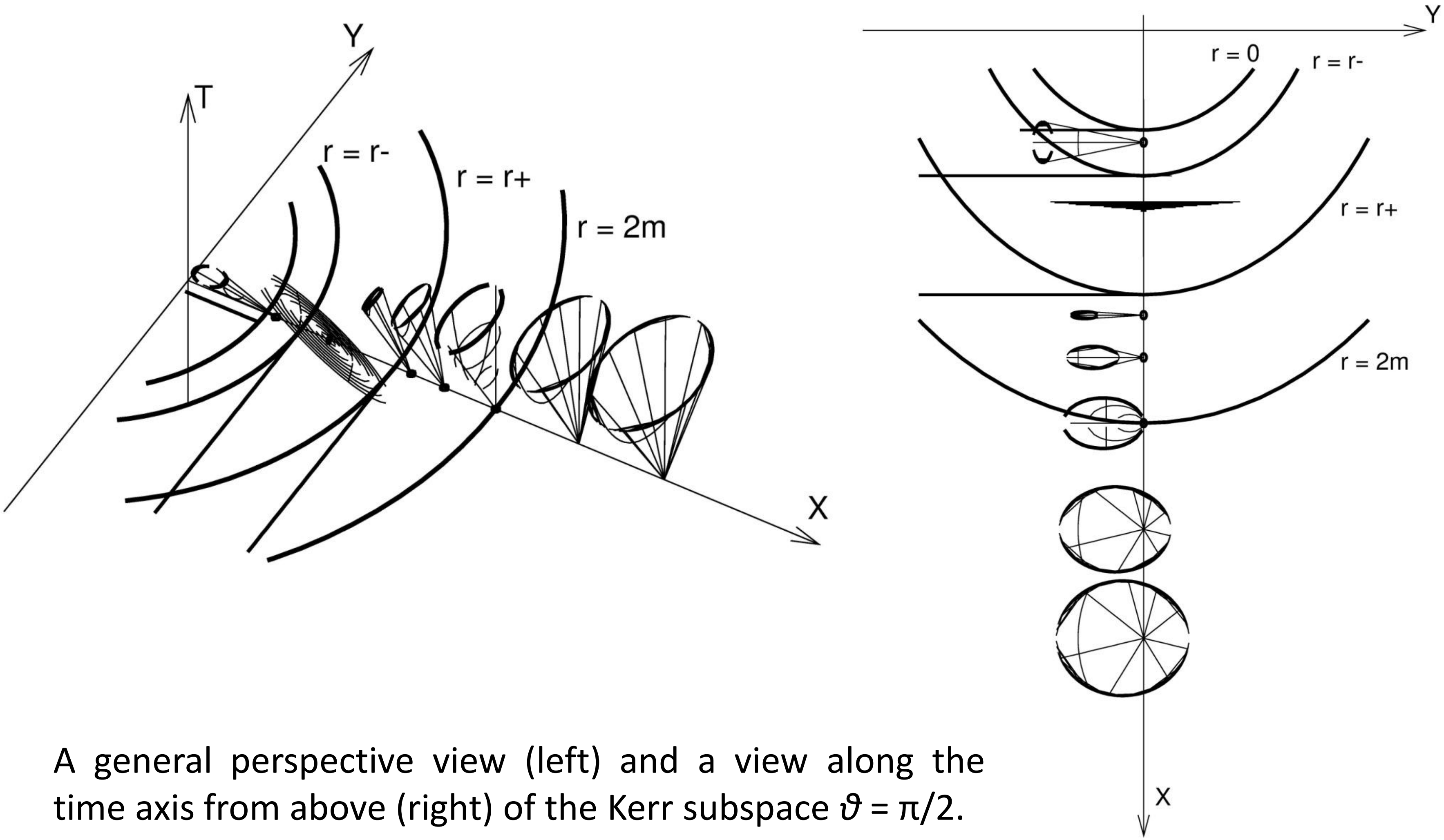

The surface still has an edge at r = 0, where it is tangent to the disc r = 0, but now the ``outer'' and the ``inner'' parts of it join smoothly at two rings (marked by dots), given by

r = m, $\vartheta$ = arc cos(m/a)

r = m, $\vartheta$ = π – arc cos(m/a).

The prevailing opinion among astrophysicists is that astronomical objects, when they are about to collapse, must get rid of the excess angular momentum to achieve $a^2 < m^2$ and avoid forming a naked singularity.

Nevertheless, the Sun and the Earth have $a^2 > m^2$ (verification of this is an exercise for the reader).




A general perspective view (left) and a view along the time axis from above (right) of the Kerr subspace $\vartheta = \pi/2$.

Large dots mark the positions of the vertices of the light cones in the T = 0 plane.

See Exercise 8 for directions on how to draw these figures.

The direction of rotation is clockwise ($a < 0$), so that the X-axis would move to the left.

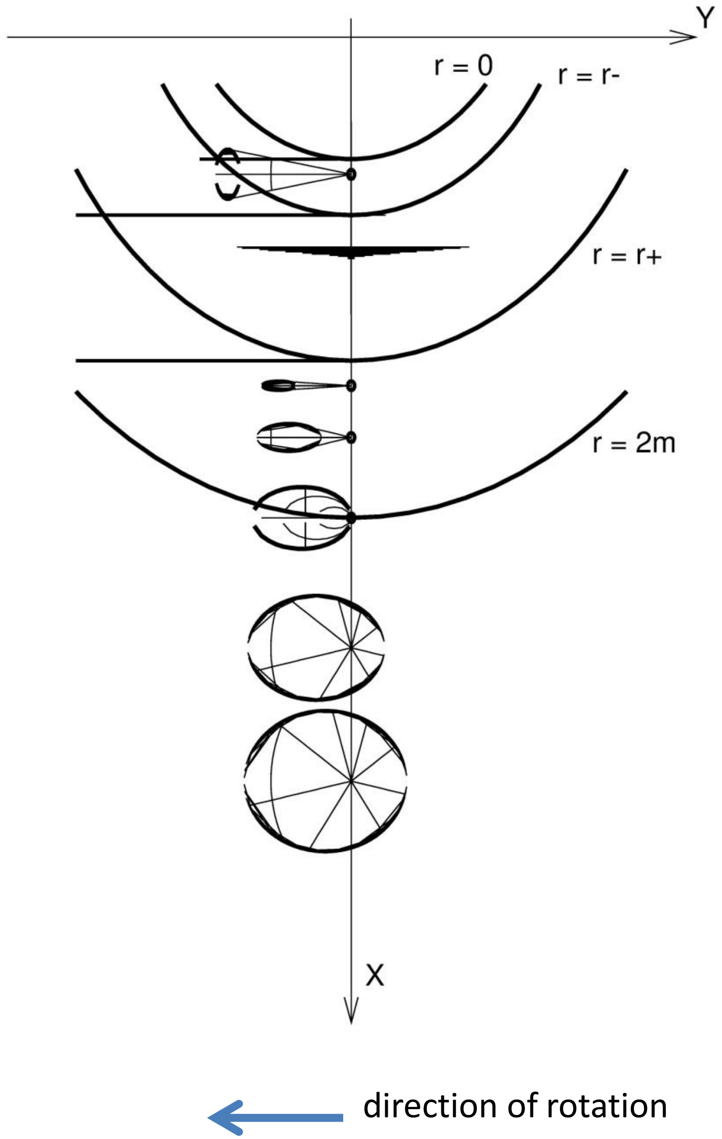
The outermost ring is the stationary limit surface at r = 2m, the two middle rings are the event horizons at $r = r_{\pm}$, and the innermost ring is the inner stationary limit surface that, because of $\vartheta = \pi/2$, coincides with the ring singularity at r = 0.

The future light cones at r >> 2m look like tilted Minkowski light cones.

At r = 2m, one generator of the light cone is parallel to the t-axis: no timelike vector at $r_+ \leq r \leq 2m$ can have a zero $\varphi$-component.

As we move from r = 2m toward $r = r_+$, the cones lean forward more and more and become thinner in all directions.

The leaning forward is a mark of ***frame dragging***.



The limit $r \rightarrow r_+$ is discontinuous.

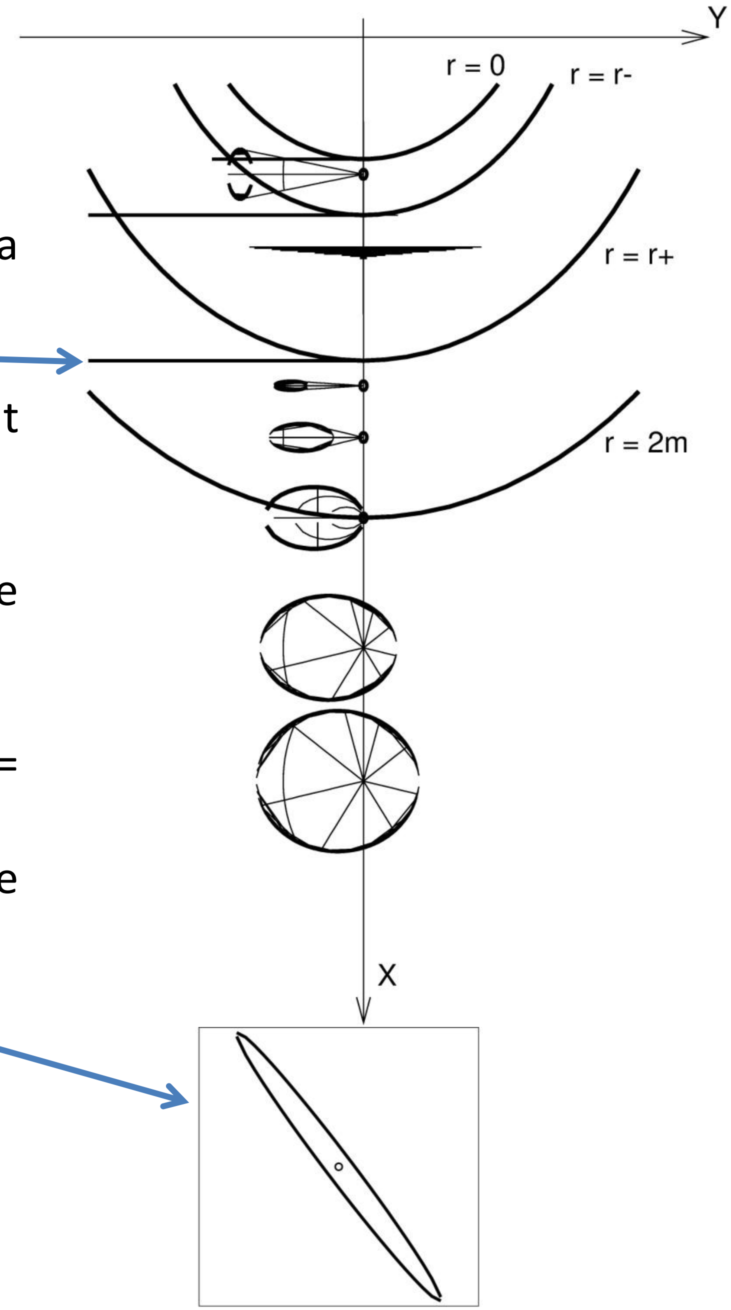
As $r \rightarrow r_+$ from the $r > r_+$ side, the cones tend to a line along the Y-direction in the $T = 0$ plane.

The intersections of the cones with a $T$ = constant plane are ellipses that recede to $Y \rightarrow -\infty$ as $r \searrow r_+$.

As $r \rightarrow r_+$ from the $r < r_+$ side, the cones tend to the whole plane $X = (r_+^{\,2} + a^2)^{1/2}$ = constant.

The intersections of the cones with the $X$ = constant planes are ellipses elongated and rotated as shown in the inset; their axes both become infinite as $r \nearrow r_+$.

A similar discontinuity exists at $r = r_-$.



Since at $r \nearrow r_+$ the cone degenerates to the plane $X = (r_+^{\ 2} + a^2)^{1/2}$ = constant; no timelike vector attached there can have a zero r-component.

In fact r takes over as the time-coordinate here.

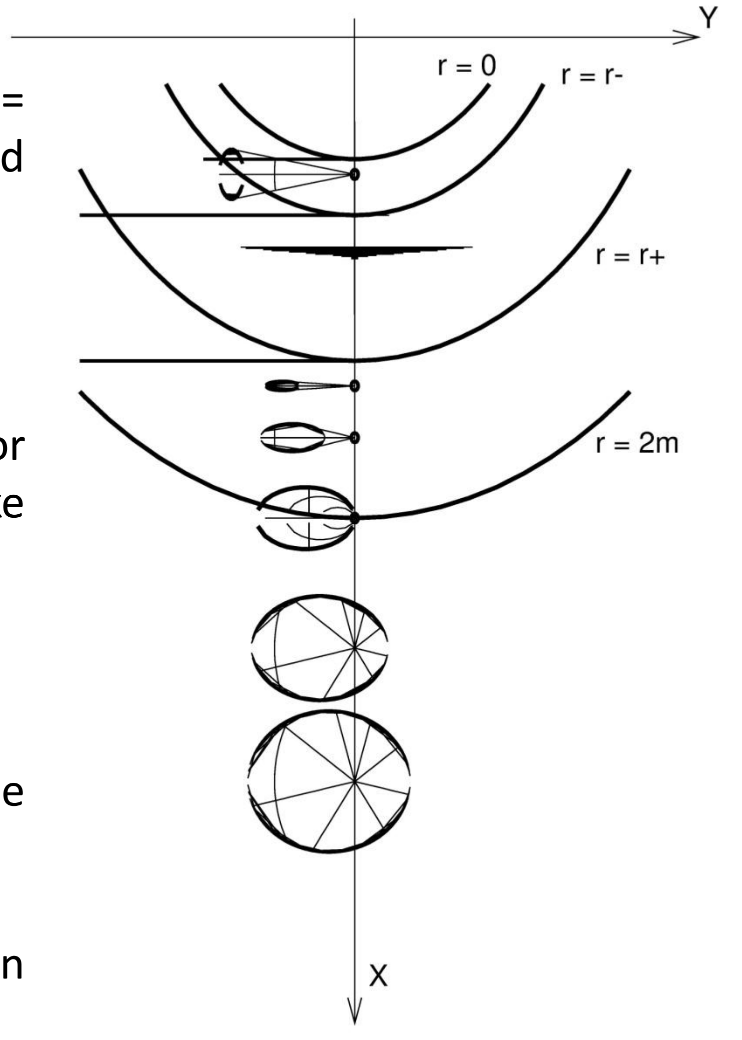
For $r_- \leq r \leq r_+$, it is impossible for an ingoing timelike or null curve to turn back without becoming spacelike along an arc or having a reflection.

This shows that $r = r_+$ is indeed an event horizon.

At $r < r_-$ the light cones are similar to those in the sector $(r_+, 2m)$.

Their intersections with a T = constant plane again become ellipses.

As $r \to 0$, they tend to the straight line $\{X = a, T = - Y\}$.



In the figures shown here, ingoing and outgoing null curves cross at $r = r_+$, but the B-L coordinates give a false picture here.

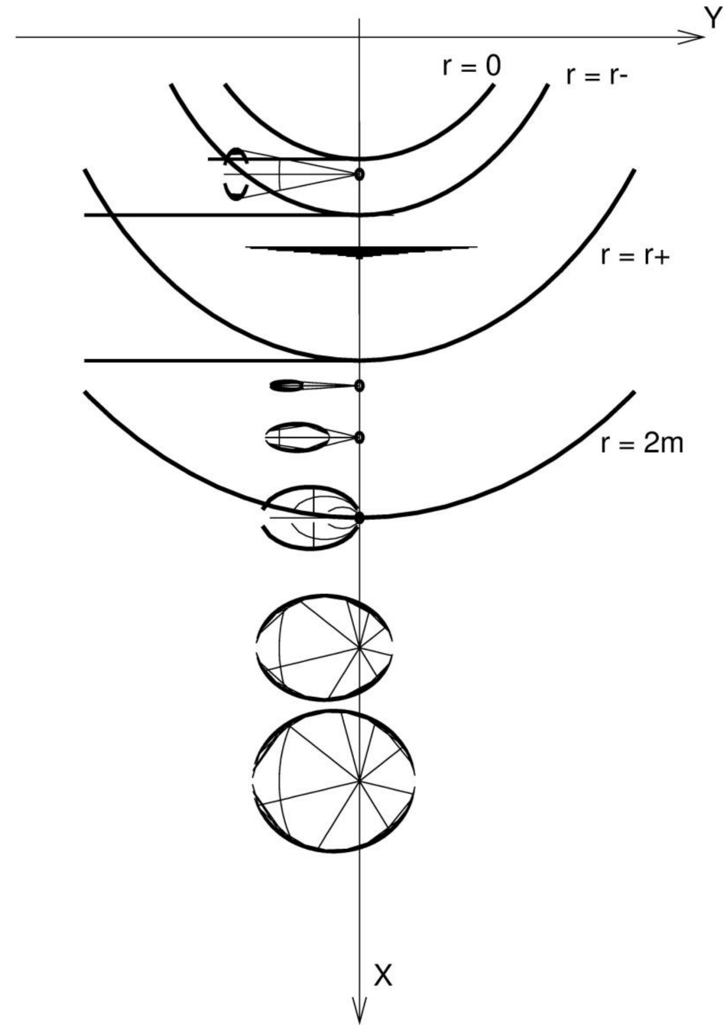
In reality, there are two event horizons at $r = r_+$, the past- and the future one.

The Kruskal diagram presents a useful analogy of the situation.

An extension of the Kerr manifold, analogous but not at all similar to that by Kruskal and Szekeres, was given by Boyer and Lindquist [191].

Important preliminary results and generalisations were presented by Carter [197,198,189].

See Ref. [11] for the presentation of the Boyer - Lindquist approach.




[197] B. Carter, Phys. Rev. 141, 1242 (1966).
[198] B. Carter, Phys. Rev. 174, 1559 (1968).
[11] J. Plebański and A. Krasiński, An introduction to general relativity and cosmology. Cambridge University Press 2006. Paperback re-edition 2012.

## 17. 4. The Hamiltonian and the Poisson bracket

When looking for an extremum of the functional

$\mathcal{L} = \int L(q^i(t),\dot{q}^i(t),t)dt$ (where $\dot{q}^i = dq^i/dt$)

with respect to the functions $q^i(t)$, i = 1, ..., m, it is often useful to re-express L as a function of 2m variables $q^i$ and $p_i = \partial L/\partial \dot{q}^i$, and then form the Hamiltonian:

$$H \stackrel{\text{def}}{=} \sum_{l=1}^{m} p_l \dot{q}^l(q,p,t) - L \tag{17.17}$$

The variables $q^i$ and $p_i$ are called *positions* and *momenta*, respectively.

In terms of H, the Euler - Lagrange equations that determine $q^i(t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0$$

become the Hamilton equations for the functions $q^i(t)$ and $p_i(t)$:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{17.18}$$

This formalism was invented for the needs of mechanics, but is useful for solving the geodesic equations in differential geometry.

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} \qquad (17.18)$$

The ***Poisson bracket*** for the functions $F(q^i, p_i)$ and $G(q^i, p_i)$, $i = 1, \ldots, m$, is defined as

$$\{F, G\} \stackrel{\text{def}}{=} \sum_{l=1}^{m} \left( \frac{\partial F}{\partial q^l} \frac{\partial G}{\partial p_l} - \frac{\partial F}{\partial p_l} \frac{\partial G}{\partial q^l} \right). \qquad (17.19)$$

From the Hamilton equations (17.18) it follows that for any function $E(q^i, p_i, t)$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{l=1}^{m} \left( \frac{\partial E}{\partial q^l} \frac{\mathrm{d}q^l}{\mathrm{d}t} + \frac{\partial E}{\partial p_l} \frac{\mathrm{d}p_l}{\mathrm{d}t} \right) + \frac{\partial E}{\partial t} = \{E, H\} + \frac{\partial E}{\partial t}. \qquad (17.20)$$

So, if E does not depend on t directly (i. e., depends on t only via $q^i$ and $p_i$) and has zero Poisson bracket with the Hamiltonian, then it is constant along the curve $\{q^i(t), p_i(t)\}$ that obeys the Hamilton equations (17.18).

## 17. 5. General geodesics

For each Killing field $k^\alpha$ there exists a first integral of the geodesic equations.

Let $p^\alpha$ be a vector field tangent to an affinely parametrised geodesic.

Then, along the geodesic,

$d(k_\alpha p^\alpha)/ds = D(k_\alpha p^\alpha)/ds = k_{\alpha;\beta}\, p^\alpha p^\beta + k_\alpha\, p^\alpha{}_{;\beta}\, p^\beta = 0,$

the first term being zero because only $k_{(\alpha;\beta)} = 0$ enters the formula, the second one being zero because $p^\alpha$ is geodesic.

For the Killing fields in the Kerr solution, $k^\alpha{}_{(t)} = \delta^\alpha{}_0$ and $k^\alpha{}_{(\varphi)} = \delta^\alpha{}_3$, there are two first integrals of the geodesic equations:

$$p_\alpha \underset{(t)}{k}{}^\alpha = p_0 \overset{\text{def}}{=} E, \qquad p_\alpha \underset{(\varphi)}{k}{}^\alpha = p_3 \overset{\text{def}}{=} -L_z \tag{17.22}$$

Moreover, there is the first integral $g_{\alpha\beta}\, p^{\alpha} p^{\beta}$ = constant that exists always. For timelike geodesics we choose the affine parameter so that

$$g_{\alpha\beta}\, p^{\alpha} p^{\beta} = \mu_0^{2}, \tag{17.23}$$

where $\mu_0$ is the mass of the orbiting particle.

Null geodesics result then as the special case $\mu_0 = 0$.

For the Kerr metric, there exists a fourth first integral, discovered by Carter [189].

The geodesic equations are the Euler - Lagrange equations for the Lagrangian

$$L = \frac{1}{2} g_{\alpha\beta} \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}s} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}s} \stackrel{\mathrm{def}}{=} \frac{1}{2} g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} \tag{17.24}$$

The momentum associated to $\dot{x}^{\alpha}$ is $p_{\alpha} = g_{\alpha\beta}\, \dot{x}^{\beta}$.

The corresponding Hamiltonian is

$$H \stackrel{\mathrm{def}}{=} p_{\alpha} \dot{x}^{\alpha} - L = \frac{1}{2} g_{\alpha\beta} \dot{x}^{\alpha} \dot{x}^{\beta} = L\left(p_{\alpha}, q^{\beta}\right) \tag{17.25}$$

The following Lemma holds:

Let the Hamiltonian have the form

$$H = \frac{1}{2} \frac{H_r + H_\mu}{U_r + U_\mu} \tag{17.26}$$

where $H_\mu$ and $U_\mu$ depend only on the coordinate $\mu$, while $H_r$ and $U_r$ depend only on r.

The functions $H_r$ and $H_\mu$ depend also on $p_\alpha$, which are independent variables on equal footing with the coordinates, but it is assumed that $H_r$ is independent of the component $p_\mu$ and $H_\mu$ is independent of the component $p_r$. Then the quantity

$$K \stackrel{\text{def}}{=} -\frac{U_r H_\mu - U_\mu H_r}{U_r + U_\mu} \tag{17.27}$$

has a vanishing Poisson bracket with the Hamiltonian, and thus is a constant of the motion.

How the Carter Hamiltonian for timelike and null geodesics in the Kerr metric is split into the four parts that appear in (17.26) and (17.27), and how the constant K is found is a horrible calculation involving horrible expressions. It is all presented in the lecture notes on the web page.

The fourth first integral of the geodesic equations found by Carter is:

$$K = a^2 \sin^2\vartheta \left[ \frac{\Sigma}{\Delta_r}\, \dot{r}^2 - \frac{\Delta_r}{\Sigma} \left( a \sin^2\vartheta \dot{\varphi} - \dot{t} \right)^2 \right]$$
$$+ \left( r^2 + a^2 \right) \left\{ \Sigma \dot{\vartheta}^2 + \frac{\sin^2\vartheta}{\Sigma} \left[ \left( r^2 + a^2 \right) \dot{\varphi} - a\dot{t} \right]^2 \right\}$$

The remaining three first integrals are:

$$E = \frac{r^2 - 2mr + a^2\cos^2\vartheta}{\Sigma}\, \dot{t} + \frac{2mra\sin^2\vartheta}{\Sigma}\, \dot{\varphi},$$
$$L_z = -\frac{2mra\sin^2\vartheta}{\Sigma}\, \dot{t} + \frac{\left[ -\Delta_r a^2 \sin^2\vartheta + \left( r^2 + a^2 \right)^2 \right] \sin^2\vartheta}{\Sigma}\, \dot{\varphi},$$
$${\mu_0}^2 = -\Sigma \left( \frac{\dot{r}^2}{\Delta_r} + \dot{\vartheta}^2 \right) - \frac{\sin^2\vartheta}{\Sigma} \left[ \left( r^2 + a^2 \right) \dot{\varphi} - a\dot{t} \right]^2 + \frac{\Delta_r}{\Sigma} \left( a \sin^2\vartheta \dot{\varphi} - \dot{t} \right)^2$$

The first integrals solved for the components of $dx^\alpha/dt$ are:

$$\begin{aligned}
\Sigma^2 \dot{r}^2 &= R(r), \\
\Sigma^2 \dot{\vartheta}^2 &= \Theta(\vartheta), \\
\Sigma \dot{\varphi} &= \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta_r}\right) L_z + \frac{2mra}{\Delta_r}\, E, \\
\Sigma \dot{t} &= -\frac{2mra}{\Delta_r}\, L_z + \left[\frac{\left(r^2+a^2\right)^2}{\Delta_r} - a^2 \sin^2\vartheta\right] E.
\end{aligned}$$

where $\Delta_r = r^2 - 2mr + a^2$, $\Sigma = r^2 + a^2 \cos^2\vartheta$, and

$$\begin{aligned}
R(r) &\stackrel{\text{def}}{=} -K\Delta_r - {\mu_0}^2\left(r^2+a^2\right)\Delta_r + \left[\left(r^2+a^2\right)E - aL_z\right]^2, \\
\Theta(\vartheta) &\stackrel{\text{def}}{=} K + {\mu_0}^2 a^2 \sin^2\vartheta - \sin^2\vartheta\left(aE - \frac{L_z}{\sin^2\vartheta}\right)^2.
\end{aligned}$$

The equation below being fulfilled all along a geodesic is a necessary and sufficient condition for the geodesic to lie in the equatorial plane $\{\vartheta = \pi/2,\ d\vartheta/dt = 0\}$:

$$K + {\mu_0}^2 a^2 - \left(aE - L_z\right)^2 = 0 \qquad (17.57)$$

see Exercise 9 in the lecture notes for the proof.

## 17. 6. Geodesics in the equatorial plane

This section is based on the papers by Bardeen [204] and Boyer and Lindquist [191].

In a spherically symmetric spacetime, each timelike or null geodesic lies in a plane, and then the coordinate axes can be chosen so that it is the equatorial plane.

In the Kerr spacetime we do not have that freedom. The equatorial plane in the BL coordinates is $\vartheta = \pi/2$, and a given geodesic either lies in it or does not.

The geodesic equations simplify a lot when it does, and we consider this case now.

[204] J. M. Bardeen, in: *Black holes -- les astres occlus*. Edited by C. de Witt and B. S. de Witt. Gordon and Breach, New York, London, Paris 1973, p. 219.

$$\Sigma^2 \dot{r}^2 = R(r) = -K\Delta_r - {\mu_0}^2 \left(r^2 + a^2\right) \Delta_r + \left[\left(r^2 + a^2\right) E - aL_z\right]^2 \qquad (17.49)$$

$\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta_r = r^2 - 2mr + a^2$.

Consider (17.49) with $\vartheta = \pi/2$, $d\vartheta/ds = 0$.

The motion can take place where $R(r) \geq 0$.

Let us treat $R(r)$ as a function of $E$. Then (17.49) becomes:

$$\frac{1}{r} R(r) = \left[r\left(r^2 + a^2\right) + 2ma^2\right] E^2 - 4amL_z E - (r - 2m){L_z}^2 - r\Delta_r {\mu_0}^2 \qquad (17.58)$$

The discriminant of $R/r$ as a function of $E$ is

$$\delta = 4r\Delta_r \left\{r{L_z}^2 + {\mu_0}^2 \left[r\left(r^2 + a^2\right) + 2ma^2\right]\right\} \qquad (17.59)$$

and for $r > 0$ the signs of $\delta$ and $\Delta_r$ are the same.

Hence, $R(r)$ has zeros only in those regions where $\Delta_r \geq 0$.

$$\frac{1}{r}\,R(r) = \left[r\left(r^2+a^2\right)+2ma^2\right]E^2 - 4amL_zE - (r-2m){L_z}^2 - r\Delta_r{\mu_0}^2 \qquad (17.58)$$

$$\delta = 4r\Delta_r\left\{r{L_z}^2 + {\mu_0}^2\left[r\left(r^2+a^2\right)+2ma^2\right]\right\} \qquad (17.59)$$

Note that $p^0 > 0$: since $p^0 \sim dx^0/ds$, the opposite would mean that the particle moves backward in time.

We find

$$p^0 = g^{00}E + g^{03}L_z = \frac{B}{\Delta_r\Sigma}\;(E-\omega L_z) \qquad (17.60)$$

where

$$B \overset{\text{def}}{=} \left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2\vartheta \geq 0, \qquad \omega \overset{\text{def}}{=} 2amr/B \qquad (17.61)$$

Hence, $p^0 > 0 \leftrightarrow E > \omega L_z$, which, in the plane $\vartheta = \pi/2$, reduces to

$$E > \frac{2am}{D}\,L_z, \qquad D \overset{\text{def}}{=} r\left(r^2+a^2\right)+2ma^2 \qquad (17.62)$$

The two zeros of $R(r) = 0$ are at

$$E_\pm = \frac{1}{D}\left[2amL_z \pm \sqrt{\Delta_r\left(r^2{L_z}^2 + {\mu_0}^2 rD\right)}\right]$$

but $E_-$ does not obey (17.62). Thus, the motion can only take place where $E > E_+ = E_{min}$.

$$r = r_\pm \overset{\text{def}}{=} m \pm \sqrt{m^2 - a^2} \qquad (17.13)$$

$$\frac{1}{r}\,R(r) = \left[r\left(r^2 + a^2\right) + 2ma^2\right]E^2 - 4amL_zE - (r - 2m){L_z}^2 - r\Delta_r{\mu_0}^2 \qquad (17.58)$$

$$\delta = 4r\Delta_r\left\{r{L_z}^2 + {\mu_0}^2\left[r\left(r^2 + a^2\right) + 2ma^2\right]\right\} \qquad (17.59)$$

If $a^2 < m^2$, then $\Delta_r > 0$ for $r < r_-$ and for $r > r_+$, where $r_\pm$ are given by (17.13).

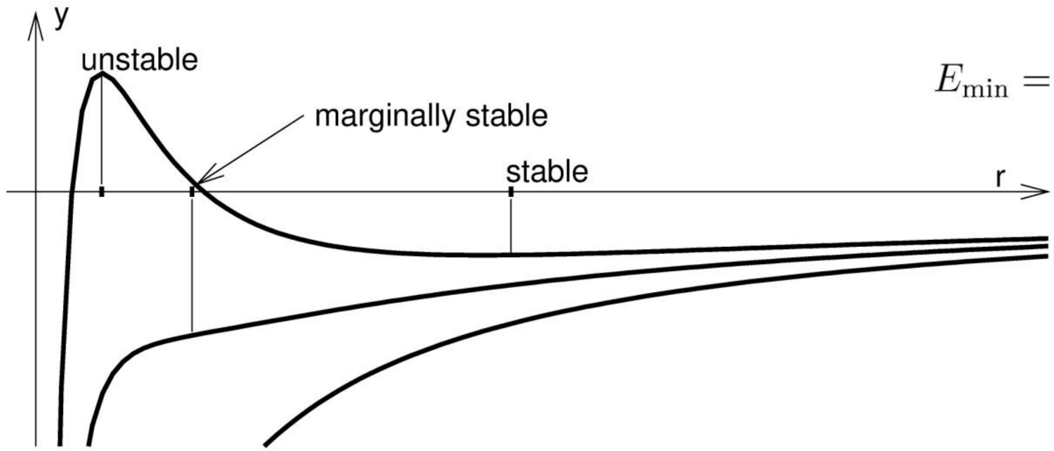
For $r_- < r < r_+$, $\delta < 0$ in (17.59), so $R(r) \neq 0$ there, and thus $dr/ds \neq 0$.

→ There exist no circular orbits and no turning points in that region.

A freely falling body that enters the region $(r_-, r_+)$ from the side of $r > r_+$ must fly all through it to the hypersurface $r = r_-$.

However, it does not have to hit the singularity, since the orbit can have a turning point at $r < r_-$.

Conversely, a body that entered the region $(r_-, r_+)$ from the side of $r < r_-$ must fly all through it to the hypersurface $r = r_+$.




$$E_{\min} = \frac{1}{D}\left[2amL_z + \sqrt{\Delta_r\left(r^2{L_z}^2 + {\mu_0}^2 rD\right)}\right]$$

$y(r) = E_{min}(r)/\mu_0 - 1$

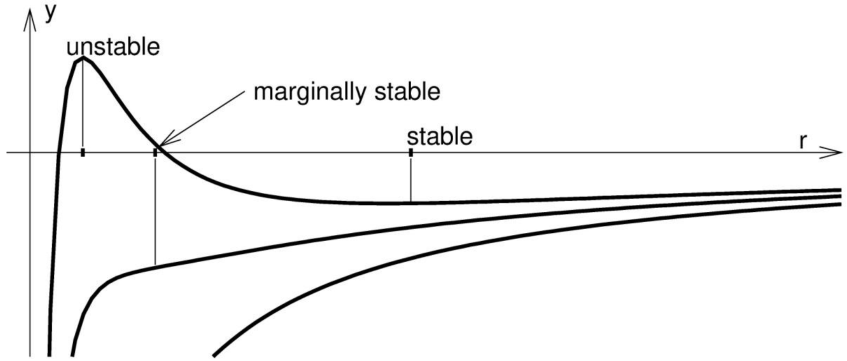
Typical graphs of the function y(r) for different values of $L_z$. The curves begin at $r = r_+$; the y-axis is drawn approximately there.

For every $L_z$, $E_{min}/\mu_0 \to 1$ as $r \to \infty$.

The lowest graph corresponds to $L_z = 0$, then y(r) is all increasing.

The uppermost graph corresponds to $|L_z|$ being large, y(r) then has a maximum that determines the position of the unstable circular orbit, and a minimum that determines the stable circular orbit.

The middle graph approximately corresponds to such $L_z$, at which the maximum and the minimum coalesce into a single point where $d^2y/dr^2 = 0$. This point determines the radius of the marginally stable circular orbit.




$$E_{\min} = \frac{1}{D}\left[2amL_z + \sqrt{\Delta_r\left(r^2{L_z}^2 + {\mu_0}^2 rD\right)}\right]$$

$y(r) = E_{min}(r)/\mu_0 - 1$

Each orbit has a constant $E \geq E_{min}(r)$.

→ The area accessible for motion is above the graph of $E_{min}(r)$, and the value of E determines the allowed range of r on the orbit.

When $L_z = 0$, $d(E_{min}/\mu_0)/dr > 0$ for all $r \geq r_+$, so $E_{min}/\mu_0 < 1$ for all $r \geq r_+$.

If $|L_z|$ is sufficiently large, then there exists an interval $(r_1, r_2)$ (where $r_+ \leq r_1 < r_2$) in which $E_{min}/\mu_0 \geq 1$, while for $r > r_2$ $E_{min}/\mu_0 < 1$, independently of the sign of $L_z$.

→ With $|L_z|$ sufficiently large, $E_{min}/\mu_0$ has a local maximum at some $r = r_u > r_+$ and a local minimum at $r = r_s > r_u$.

The region where $E_{min}/\mu_0 < 1$ is the locus of bound orbits, and $r = r_s$ is the radius of the stable circular orbit (i.e. there exists one for each sufficiently large value of $|L_z|$, for each sign of $L_z$). A circular orbit also exists at $r = r_u$, but is unstable.

$$E_{\min} = \frac{1}{D}\left[2amL_z + \sqrt{\Delta_r\left(r^2{L_z}^2 + {\mu_0}^2 rD\right)}\right] \qquad (17.63)$$

$L_z$ is the orbital angular momentum of the body on the geodesic, while $a$ is the angular momentum per unit mass of the source of the gravitational field.

When $aL_z > 0$, the two angular momenta are parallel, such orbits are called ***direct***.

When $aL_z < 0$, they are antiparallel, such orbits are called ***retrograde***.

$E_{min}$ in (17.63) at a given r is different for each of these orbits.

This difference is a relativistic effect: in Newton's theory the two orbits are the same.

The difference between them in relativity is quite pronounced.

Note from (17.63) that with $aL_z < 0$ and r sufficiently close to $r_+$ (i.e. with $\Delta_r$ close to zero) $E_{min} < 0$, so E can be negative.

If $E > 0$, then it is the rest energy of the particle ``at infinity''.

→ $E < 0$ means that the particle lost more than its rest energy to enter such an orbit. This effect does not appear for $aL_z > 0$, or in the Schwarzschild limit $a = 0$.

$$\mathrm{d}s^2 = \left(1-\frac{2mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{4mra\sin^2\vartheta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma}+r^2+a^2\right)\sin^2\vartheta\mathrm{d}\varphi^2 - \frac{\Sigma}{\Delta_r}\mathrm{d}r^2 - \Sigma\mathrm{d}\vartheta^2. \quad (17.12)$$

$\Sigma = r^2 + a^2\cos^2\vartheta, \quad \Delta_r = r^2 - 2mr + a^2$

$$p^0 = g^{00}E + g^{03}L_z = \frac{B}{\Delta_r\Sigma}\ (E-\omega L_z) \quad (17.60)$$

$$p_\alpha \underset{(t)}{k}{}^\alpha = p_0 \overset{\text{def}}{=} E, \quad p_\alpha \underset{(\varphi)}{k}{}^\alpha = p_3 \overset{\text{def}}{=} -L_z \quad (17.22)$$

$$B \overset{\text{def}}{=} \left(r^2+a^2\right)^2 - \Delta_r a^2\sin^2\vartheta \geq 0, \qquad \omega \overset{\text{def}}{=} 2amr/B \quad (17.61)$$

Where can negative-energy orbits exist? To verify, let us take the Kerr metric in the BL coordinates, (17.12), then let us choose at each point of the spacetime the following orthonormal vector basis $e^i{}_\alpha$:

$$e^0{}_\alpha\mathrm{d}x^\alpha = \mathrm{e}^\nu\mathrm{d}t, \qquad e^1{}_\alpha\mathrm{d}x^\alpha = \mathrm{e}^\lambda\mathrm{d}r, \qquad e^2{}_\alpha\mathrm{d}x^\alpha = \sqrt{\Sigma}\mathrm{d}\vartheta, \qquad e^3{}_\alpha\mathrm{d}x^\alpha = \mathrm{e}^\psi(\mathrm{d}\varphi - \omega\mathrm{d}t), \quad (17.64)$$

where B and ωwere defined in (17.61), and

$$\mathrm{e}^{2\nu} = \frac{\Delta_r\Sigma}{B}, \qquad \mathrm{e}^{2\lambda} = \frac{\Sigma}{\Delta_r}, \qquad \mathrm{e}^{2\psi} = \frac{B}{\Sigma}\sin^2\vartheta \quad (17.65)$$

The projections of the momentum defined in (17.22) on $e^0{}_\alpha$ and $e^3{}_\alpha$ are then

$$p_{\widehat{0}} = \mathrm{e}^{-\nu}\left(E-\omega L_z\right), \qquad p_{\widehat{3}} = \mathrm{e}^{-\psi}p_3 = -\mathrm{e}^{-\psi}L_z \quad (17.66)$$

In consequence of $p^0 > 0$ and (17.60), $p_{\hat{0}} = p^{\hat{0}} > 0$ where $\Delta_r > 0$.

$p_{\hat{0}} = \mathrm{e}^{-\nu}(E - \omega L_z)\,, \qquad p_{\hat{3}} = \mathrm{e}^{-\psi} p_3 = -\mathrm{e}^{-\psi} L_z$ (17.66) $B \stackrel{\text{def}}{=} (r^2 + a^2)^2 - \Delta_r a^2 \sin^2\vartheta \geq 0, \qquad \omega \stackrel{\text{def}}{=} 2amr/B$

The $p_i$ obey

$$p_{\hat{0}}{}^2 - p_{\hat{1}}{}^2 - p_{\hat{2}}{}^2 - p_{\hat{3}}{}^2 = \mu_0{}^2 > 0 \Longrightarrow |p_{\hat{3}}| < p_{\hat{0}} \tag{17.67}$$

From (17.66), $E = \mathrm{e}^{\nu} p_{\hat{0}} - \omega \mathrm{e}^{\psi} p_{\hat{3}}$.

→ $E < 0$ implies $\omega p_{\hat{3}} > 0$ (this is consistent with $aL_z < 0$).

So, using (17.67), $E < 0$ implies $e^{\nu} p_{\hat{0}} < e^{\psi} |\omega| \, | p_{\hat{3}} | < e^{\psi} |\omega| p_{\hat{0}}$.

It follows that $\omega^2 e^{2\psi} > e^{2\nu}$, which means $g_{tt} < 0$.

→ Orbits with negative energy can exist between the stationary limit hypersurfaces.

This was a necessary condition for the existence of orbits with $E < 0$.

A sufficient condition is that the body is between the stationary limit hypersurfaces and the component $p_r = 0$. The proof is given in the notes.

## 17.7. The Penrose process

Penrose [205] contemplated a process by which *in principle* the rotational energy of a Kerr black hole can be extracted.

The idea is based on the observation we made earlier that a body on a retrograde orbit inside the outer stationary limit hypersurface (OSLH) can have a negative energy if it is close enough to the event horizon $r = r_+$.

In brief, Penrose's idea was this: put two masses at the ends of a spring, then squeeze the spring and bind the masses together.

Then send the composite on an orbit that enters the region between the OSLH and $r = r_+$, called ***ergosphere***.

(The name comes from the Greek word **εργο**, meaning ``work'' -- because, as shown below, a rotating black hole is in principle able to do some work by extracting energy from the ergosphere.)

Design the orbit so that it has its turning point close to $r = r_+$ (how close will become clear below).

[205] R. Penrose, in *Riv. Nuovo Cimento, Numero Speziale* **I**, 252 (1969); reprinted in *Gen. Relativ. Gravit.* **34**, 1141 (2002), with an editorial note by A. Kr\'olak, *Gen. Relativ. Gravit.* **34**, 1135 (2002) and author's biography by W. Israel, *Gen. Relativ. Gravit.* **34**, 1138 (2002).

When the composite object is at the turning point (where $dr/ds = 0$), release the spring in such a direction that one of the masses is sent, with initial $dr/ds = 0$, on a retrograde orbit with $E < 0$ and with $|L_z|$ sufficiently small for it to fall through the event horizon.

It follows from the reasoning presented earlier that this is possible: we direct the ejected mass so that $aL_z < 0$, we make $|L_z|$ small enough that the mass is sure to go through $r = r_+$, and the orbit has to be pre-designed so that at the turning point $e^{\nu} p_{\hat{0}} < |\omega L_z|$.

Since the mass dropped into the black hole carried away some negative energy, the other mass acquires some additional energy and additional momentum by recoil.

Consequently, it will return to the outside of the OSLH having a greater energy than it had at the beginning of the journey.

This trick can be applied for as long as the OSLH exists.

The logical conclusion is that the extra energy carried away by the returning mass was gained at the expense of the rotational energy of the black hole: the slower the b. h. rotates, the smaller $|a|$ becomes, and the smaller the volume of the ergosphere.

However, this is a speculation that goes beyond the area of applicability of the Kerr metric.

In order to discuss this energy extraction process in a correct way, we would have to use a nonstationary solution in which the angular momentum of the rotating body can depend on time.

Papers are being published on the observational signatures of the Penrose process going on in real astronomical objects, but the topic is treated as non-mainstream.

$$\begin{aligned} ds^2 &= \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4mra\sin^2\vartheta}{\Sigma} dt d\varphi - \left(\frac{2mra^2\sin^2\vartheta}{\Sigma} + r^2 + a^2\right) \sin^2\vartheta d\varphi^2 \\ &- \frac{\Sigma}{\Delta_r} dr^2 - \Sigma d\vartheta^2. \end{aligned} \tag{17.12}$$

## 17.8. Stationary-axisymmetric spacetimes and locally non-rotating observers

A spacetime is ***stationary*** when it has a timelike Killing field $k_{(0)}{}^{\alpha}$ (if $k_{(0)}{}^{\alpha}$ is hypersurface- orthogonal, then the spacetime is ***static***).

It is ***axisymmetric*** when it has a Killing field $k_{(3)}{}^{\alpha}$ whose integral lines are closed.

For ***stationary-axisymmetric*** spacetimes it is assumed in addition that the two Killing fields commute.

The Kerr metric is an example.

Then, coordinates can be chosen so that the metric is independent of $x^0 = t$ (where $dx^{\alpha}/dt = k_{(0)}{}^{\alpha}$) and of $x^3 = \varphi$ (where $dx^{\alpha}/d\varphi = k_{(3)}{}^{\alpha}$).

Let the other two coordinates be $x^1$ and $x^2$.

It is also assumed that the surfaces generated by the Killing fields admit orthogonal surfaces, i.e. that in the coordinates adapted to the Killing fields
$g_{01} = g_{02} = g_{13} = g_{23} = 0$.
This property is called ***orthogonal transitivity***.

In these coordinates the metric, and the motion of matter if any is present, is invariant under the discrete transformation $(t, \varphi) \rightarrow (-t, -\varphi)$.

(An example of a configuration that does not obey this is a rotating gaseous body, inside which the gas circulates in the meridional planes.)

Several theorems were proven in which $[k_{(0)}, k_{(3)}] = 0$ and orthogonal transitivity follow from other assumptions.

Their intention was to show that spacetimes that do not possess these properties are rare or unimportant or weird.

The fact is, though, that not much is known about the cases left out.

In a stationary-axisymmetric spacetime that is orthogonally transitive, coordinates in the ($x^1$, $x^2$) surfaces can be chosen so that $g_{12} = 0$. The metric is thus:

$$\mathrm{d}s^2 = g_{00}\mathrm{d}t^2 + 2g_{03}\mathrm{d}t\mathrm{d}\varphi + g_{33}\mathrm{d}\varphi^2 + g_{11}\mathrm{d}(x^1)^2 + g_{22}\mathrm{d}(x^2)^2 \qquad (17.75)$$

When nothing else is assumed about the two Killing fields, the basis of their space can be chosen arbitrarily. The transformation of the basis

$$\underset{(0)}{k}{}' = C_0 \underset{(0)}{k} + D_0 \underset{(3)}{k}, \qquad \underset{(3)}{k}{}' = C_3 \underset{(0)}{k} + D_3 \underset{(3)}{k}$$

induces a transformation of the adapted coordinates; the ($t'$, $\varphi'$) adapted to $k'_{(0)}$ and $k'_{(3)}$ are

$$t' = C_0 t + C_3\varphi, \qquad \varphi' = D_0 t + D_3\varphi \qquad (17.76)$$

The transformed $g_{\alpha\beta}$ is still independent of t and $\varphi$, and is orthogonally transitive, only $g_{00}$, $g_{03}$ and $g_{33}$ are reshuffled among themselves.

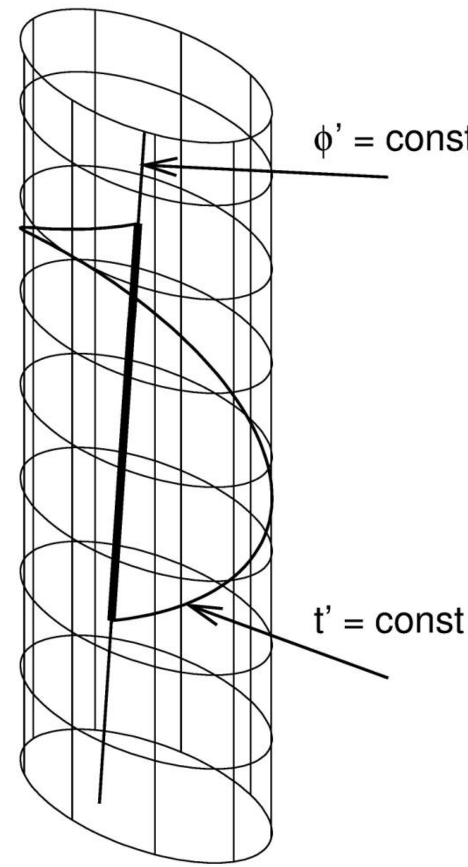
$$t' = C_0 t + C_3 \varphi, \qquad \varphi' = D_0 t + D_3 \varphi$$

(17.76)

But the Killing fields $k_{(0)}$ and $k_{(3)}$ are *unique* if the spacetime is asymptotically flat.

Then, at infinity, the integral lines of $k_{(3)}$ are closed, and $\varphi$ is periodic with period $2\pi$.

This excludes those transformations (17.76) in which $C_3 \neq 0$, or else the strange behaviour illustrated in the figure would occur: after increasing $\varphi$ by $2\pi C_3 D_0/(C_0 D_3 - C_3 D_0)$ we would land at the same t'-line from which we started, but with the t' coordinate increased by $\Delta t' = - 2\pi C_3$.




The orbits of the $k_{(3)}$ field would thereby be changed into infinite helices, while on two sides of the initial t'-line adjacent points would exist whose time-coordinate would differ by $\Delta t'$.

(However, exactly this kind of time coordinate is used on the surface of the Earth -- the discontinuity occurs across the line of change of date that runs through the middle of the Pacific.)

$$t' = C_0 t + C_3\varphi, \qquad \varphi' = D_0 t + D_3\varphi \qquad (17.76)$$

The constant $D_0$ has to be zero for a different reason.

Suppose the coordinates of the asymptotically flat spacetime go over, at infinity, into the cylindrical coordinates in which

$$ds^2 = dt^2 - r^2 d\varphi^2 - dr^2 - dz^2. \qquad (17.77)$$

This means that asymptotically $g_{03}$ and $g_{33}/r^2$ must go to zero and to -1, respectively.

After a transformation (17.76) with $D_0 \neq 0$, the $g_{03}$ component would become

$$g'_{03} = (g_{03} - D_0 g_{33}/D_3)/(C_0 D_3),$$

and would be nonzero at infinity. So, $D_0 = 0$ in consequence of asymptotic flatness.

Finally, $D_3 = 1$ in order that the period of $\varphi'$ is $2\pi$.

The only remaining freedom is the choice of the unit of time, connected with $C_0$.

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\varphi + g_{33}d\varphi^2 + g_{11}d(x^1)^2 + g_{22}d(x^2)^2 \qquad (17.75)$$

Because of this uniqueness, ***locally nonrotating observers*** exist [206].

Write (17.75) as

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\lambda}dr^2 - e^{2\mu}d\vartheta^2 \qquad (17.78)$$

Now imagine an observer moving with four-velocity $U^\alpha$ such that $U^r = U^\vartheta = 0$, i.e. circulating within the ($t$, $\varphi$) surface.

Define $\Omega = U^\varphi/U^t = d\varphi/dt$ - this is the angular velocity of the observer.

Imagine that she has set up mirrors all along her orbit so that light rays can travel around the same circle in both directions.

Let the time of the round-trip of the light ray be $T_\varepsilon$, with $\varepsilon = +1$ for the ray rotating forward, and $\varepsilon = -1$ for the ray rotating backward with respect to the observer.

[206] J. M. Bardeen, *Astrophys. J.* **162**, 71 (1970).

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu}\mathrm{d}t^2 - \mathrm{e}^{2\psi}\left(\mathrm{d}\varphi - \omega \mathrm{d}t\right)^2 - \mathrm{e}^{2\lambda}\mathrm{d}r^2 - \mathrm{e}^{2\mu}\mathrm{d}\vartheta^2 \qquad (17.78)$$

The observer will during the time $T_\varepsilon$ move from $\varphi_1$ to $\varphi_2 = \varphi_1 + \Omega T_\varepsilon$.

The light ray, during its round trip, will move from $\varphi_1$ to $\varphi_3 = \varphi_1 + \Omega T_\varepsilon + 2\pi\varepsilon$.

Along the ray $ds^2 = 0$, so $\varepsilon e^\nu dt = e^\psi (d\varphi - \omega dt)$, and consequently

$$\mathrm{d}t = \frac{1}{\omega + \varepsilon \mathrm{e}^{\nu-\psi}}\,\mathrm{d}\varphi \qquad (17.79)$$

Integrating this over the whole round-trip we get

$$T_\varepsilon = \frac{\Omega T_\varepsilon + 2\pi\varepsilon}{\omega + \varepsilon \mathrm{e}^{\nu-\psi}} \qquad (17.80)$$

Solving this for $T_\varepsilon$ we obtain

$$T_\varepsilon = \frac{2\pi}{\mathrm{e}^{\nu-\psi} - \varepsilon(\Omega - \omega)} \qquad (17.81)$$

$$ds^2 = e^{2\nu}dt^2 - e^{2\psi}(d\varphi - \omega dt)^2 - e^{2\lambda}dr^2 - e^{2\mu}d\vartheta^2 \qquad (17.78)$$

$$T_\varepsilon = \frac{2\pi}{e^{\nu-\psi} - \varepsilon(\Omega - \omega)} \qquad (17.81)$$

Along the observer's path $d\varphi = \Omega dt$, so her proper time s is related to t by

$$ds^2 = \left[e^{2\nu} - e^{2\psi}(\Omega - \omega)^2\right] dt^2 \qquad (17.82)$$

Hence, the proper time of the round-trip of the ray measured by the observer is

$$S_\varepsilon = \sqrt{e^{2\nu} - e^{2\psi}(\Omega - \omega)^2}\,T_\varepsilon \equiv 2\pi e^{\psi}\sqrt{\frac{1 + \varepsilon e^{\psi-\nu}(\Omega - \omega)}{1 - \varepsilon e^{\psi-\nu}(\Omega - \omega)}} \qquad (17.83)$$

This time is different for the forward ray and for the backward ray ($S_1 \neq S_{-1}$) except when $\Omega = \omega$.

In an asymptotically flat spacetime the (t, $\varphi$) coordinates of (17.78) are unique, so $\omega$ is uniquely determined by the geometry, and so ***there is a uniquely determined set of observers for whom the effect of rotation, ($S_1 \neq S_{-1}$), disappears***.

$$\mathrm{d}s^2 = \mathrm{e}^{2\nu}\mathrm{d}t^2 - \mathrm{e}^{2\psi}\left(\mathrm{d}\varphi - \omega\mathrm{d}t\right)^2 - \mathrm{e}^{2\lambda}\mathrm{d}r^2 - \mathrm{e}^{2\mu}\mathrm{d}\vartheta^2 \tag{17.78}$$

These privileged observers are called ***locally nonrotating observers*** [206].

Their worldlines are orthogonal to the t = constant hypersurfaces, and so their rotation tensor is zero.

The quantities present in (17.78) were defined for the Kerr metric in eqs. (17.61) and (17.65).

The Kerr metric is stationary, axisymmetric and asymptotically flat, with the period of $\varphi$ equal to 2π, so it allows the existence of locally nonrotating observers.

## 17.9. A source of the Kerr field?

The Kerr solution has been known for more than 50 years now, and from the beginning it provoked the question: what material body could generate such a vacuum field around it?

Several authors tried to find a model of the source, but so far without success.

The most promising result is that of Roos [207].

He investigated the existence of perfect fluid sources for stationary axisymmetric vacuum metrics.

In application to the Kerr metric, he proved that the Einstein equations in the source are integrable in a finite neighbourhood of the hypersurface given by
$r = r_0\,(1 + \sin^2 \vartheta)$ ,
where $r_0 > r_+$, in the BL coordinates, but there exist no solutions if the boundary hypersurface is $r = r_0 =$ constant.

[207] W. Roos, Gen. Relativ. Gravit. 7, 431 (1976).

All the attempts to find a reasonable explicit solution failed so far.

The sources known are all rather artificial (see the notes on the web page for a reference): a 2-dimensional disc spanned on the singular ring of the Kerr metric, a body with anisotropic stresses, sometimes enveloped in a crust of another kind of anisotropic matter.

The continuing lack of success prompted some authors to suggest that a perfect fluid source might not exist.

The opinion of this author is that a bright new idea is needed, as opposed to routine standard tricks tested so far.

It is enough to have a look at the Newtonian models of rotating bodies that, even in the apparently simple cases of homogeneous density distribution, are really tricky [209] to realise that the corresponding problem in relativity must be at least as difficult.

[209] S. Chandrasekhar, Ellipsoidal figures of equilibrium. Yale University Press, New Haven and London 1969, p. 46.