Part II

THE GRAVITATION THEORY

Chapter 10

The Einstein equations and the sources of a gravitational field.

10.1 Why Riemannian geometry?

As argued in Sec. 1.4, gravitational forces can be simulated by inertial forces in accelerated motion. Special relativity describes relations between objects in uniform motion with respect to inertial frames, while gravitational interactions are neglected. The metric of the Minkowski spacetime in an inertial reference frame has constant coefficients. If we transform that metric to an accelerated frame, the components of the metric will become functions. Hence, a gravitational field should have the same effect: the metric should have non-constant components – but, unlike in the Minkowski spacetime, it should not be possible to make the metric components constant by a coordinate transformation. This was, in great abbreviation, the basic observation that led Einstein to general relativity [48].

This idea had to be supplemented with equations that would generalise Newton's laws of gravitation, and would relate the metric to a measure of the gravitational field. The derivation of these equations will be presented in the present chapter.

10.2 Local inertial frames.

Let us recall the conclusion of chapter 1: the Universe is permeated by gravitational fields that cannot be screened. Their intensity can be decreased by going away from the sources, but one can never decrease that intensity below the minimum determined by the local mean density of matter in the Universe. For this reason, no body in the Universe moves freely in the sense of Newton's mechanics, and consequently inertial frames can be realised only with a limited precision. Moreover, there exists no natural standard of a straight line, so the departures of real motions from rectilinearity cannot be measured.

However, let us recall that for a body falling freely in a gravitational field the inertial

[48] A. Einstein, *Ann. Physik* **49**, 769 (1916); English translation: pp. 109-164 in Ref. [6].

force caused by the acceleration balances the gravitational force. Assume for the beginning that the gravitational field is homogeneous. Then, two bodies falling freely in it will have the same acceleration all the time, so their relative acceleration will be zero, and, relative to each other, they will either be at rest or move with a constant velocity. Consequently, the frame of reference connected with a body falling freely in a homogeneous gravitational field is inertial.

But homogeneous gravitational fields do not exist in Nature. However, at every point along the trajectory of a body falling freely in any real gravitational field, the gravitational force and the inertial force cancel each other. Hence, if the gravitational field is continuous (which is the case in all practical instances), then, in a *sufficiently small* neighbourhood of the falling body the inertial forces will be arbitrarily small. Given the minimum force ε that can be measured by a given apparatus, a falling body B will, at every point of its trajectory, determine a sphere of radius δ inside which the inertial force will be smaller than ε, i.e. non-measureable. Inside that sphere the reference system defined by B will thus be "practically" inertial. It is called the **local inertial frame** of the body B – *local* because it differs from the universal inertial frame postulated in Newton's theory. There is an infinity of local inertial frames and two different such frames will in general move with acceleration relative to each other – for example, the local inertial frames of bodies falling freely toward the Earth from opposite directions. Thus, at a sufficiently large distance from a freely falling body its local inertial frame ceases to be inertial.

10.3 Trajectories of free motion in Einstein's theory.

Let us recall one more conclusion of chapter 1: Since no model of a straight line exists in Nature, it will be simpler to assume that the geometry of our space is non-Euclidean, and in that geometry material bodies move by free motion. What was called "gravitational field" in Newton's theory will be a consequence of the non-Euclidean geometry in which the motions take place. By the argument of Section 10.1, the appropriate class of geometries are the Riemann geometries. They do contain the Euclidean geometry as a special case, and, because of the coordinate-independent formalism they use, they are consistent with the postulate of equivalence of all reference systems.

In a Riemann space, the analogue of a straight line is a geodesic line. If the curvature goes to zero, the Riemannian geometry goes over into the Minkowski geometry, and the geodesic lines go over into straight lines. Hence, geodesic lines are natural candidates for the trajectories of free motion. In addition, they have the following property:

Theorem 10.1 *For a given timelike geodesic G, the coordinates in a Riemann space V_4 can be chosen so that the Christoffel symbols vanish along G.*

The vanishing of the Christoffel symbols means that along the geodesic, the local gravitational field will be approximately zero (not exactly zero because the derivatives of the Christoffel symbols will not vanish, and so the curvature will be nonzero.)

Proof:

(Latin indices will label vectors, the lower case ones will run through the values $0, 1, 2, 3$, the upper case ones will run through the values $1, 2, 3$. Tensor indices running through the values $1, 2, 3$ will also be denoted by Latin letters. Wherever confusion might arise, vector indices will have a hat over them.)

At a chosen point $p_0 \in G$, we choose such a basis in the tangent space that $e_{\widehat{0}}{}^{\alpha}(p_0)$ is tangent to G, and the other vectors, $e_A{}^{\alpha}(p_0)$, $A = 1, 2, 3$ are orthogonal to $e_{\widehat{0}}{}^{\alpha}(p_0)$, i.e. $g_{\alpha\beta} e_{\widehat{0}}{}^{\alpha} e_A{}^{\alpha}\big|_{p_0} = 0$. Then we define the bases $e_i{}^{\alpha}(p)$ in a neighbourhood of G as follows:

1. For $p \in G$, we transport $e_i{}^{\alpha}(p_0)$ parallely from p_0 to p along G.

2. For $p \notin G$ we draw a geodesic G'_p through p that intersects G orthogonally (see Fig. 10.1). Let $p' \in G$ be the point of intersection of G'_p and G. Then we transport the basis $e_i{}^{\alpha}(p')$ (already defined in point 1) to p along G'_p.

This procedure works provided there are no singular points on G'_p (otherwise the geodesic would not go through the singularity). It gives a unique result provided that p is not too distant from G, otherwise the geodesics orthogonal to G might intersect each other.

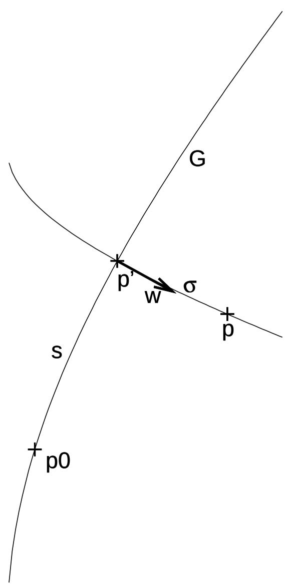

Figure 10.1: Construction of the Fermi coordinates in which the Christoffel symbols vanish along a given timelike geodesic G. More explanation in the text.

The following equations hold on G:

$$e_i{}^{\alpha}{}_{;\beta}\big|_G = 0, \qquad e^i{}_{\alpha;\beta}\big|_G = 0. \tag{10.1}$$

The first eq. follows because (i) $e_i{}^{\alpha}{}_{;\mu}\, e_{\widehat{0}}{}^{\mu}\big|_G = 0$ in consequence of the basis $e_i{}^{\alpha}$ being transported parallely along G and $e_{\widehat{0}}{}^{\mu}$ being tangent to G; (ii) $e_i{}^{\alpha}{}_{;\mu}\, e_S{}^{\mu}\big|_G = 0$ $(S = 1, 2, 3)$ in consequence of $e_S{}^{\mu}$ at G being tangent to one of the geodesics used to transplant the bases parallely to other points. Consequently, $e_i{}^{\alpha}{}_{;\mu}\, e_j{}^{\mu}\big|_G = 0$ for all $j = 0, 1, 2, 3$. The second of (10.1) follows from this one and $e_i{}^{\alpha} e^i{}_{\beta} = \delta^{\alpha}{}_{\beta}$.

Now using (10.1) in the definition of the connection coefficients, eq. (4.19), we get

$$\left.\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\right|_G = e_s{}^\alpha e^s{}_{\beta,\gamma}|_G\,. \tag{10.2}$$

Since the Christoffel symbols are symmetric, this implies at once

$$e^i{}_{\beta,\gamma}\big|_G = e^i{}_{\gamma,\beta}\big|_G\,. \tag{10.3}$$

We have defined the basis (up to rotations of $e_A{}^\alpha(p_0)$, $A = 1, 2, 3$), but we have not so far defined the coordinates. Let w^α be the unit tangent vector to G'_p ($w^\alpha w_\alpha = 1$, see Fig. 10.1), and let σ be the length of the segment of G'_p between p' and p ($\sigma = 0$ for points on G). Let s be the length of the segment of G between p_0 and p'. For the point p we define the time coordinate $x^0 = s$ and the space coordinates

$$x^A = \sigma w^\rho(p')e^{\widehat{A}}{}_\rho(p'), \qquad A = 1, 2, 3. \tag{10.4}$$

In the coordinates (s, x^A) thus defined, called **Fermi coordinates**, we have

$$e_{\widehat{0}}{}^\alpha(p') = \left.\frac{\partial x^\alpha}{\partial s}\right|_G \underset{*}{=} \delta^\alpha{}_0, \tag{10.5}$$

since $e_{\widehat{0}}{}^\alpha$ is tangent to G, and

$$e_A{}^0(p') \underset{*}{=} 0, \qquad A = 1, 2, 3, \qquad w^0 \underset{*}{=} 0 \tag{10.6}$$

because $e_A{}^\alpha(p')$ and w^α lie in the subspace of the tangent space to V_4 in which $x^0 =$ constant. Eqs. (10.5) and (10.6) imply that the matrix $e_i{}^\alpha(p')$ has a block form. Consequently, the inverse matrix $e^i{}_\alpha(p')$ must have the same block form:

$$e^{\widehat{0}}{}_A(p') \underset{*}{=} \delta^0{}_\alpha, \qquad e^{\widehat{A}}{}_0(p') \underset{*}{=} 0. \tag{10.7}$$

Now let us differentiate (10.4) by σ. Since $w^\rho(p')$ and $e^{\widehat{A}}{}_\rho(p')$ do not depend on σ, we have

$$w^A(\sigma) \overset{\text{def}}{=} \frac{\mathrm{d}x^A}{\mathrm{d}\sigma} = w^\rho(p')e^{\widehat{A}}{}_\rho(p') \underset{*}{=} w^A(p'), \tag{10.8}$$

which means that w^A does not depend on σ. In consequence of (10.6), this implies that $w^A(p') \underset{*}{=} w^K(p')e^{\widehat{A}}{}_K(p')$, $K = 1, 2, 3$, and so

$$e^{\widehat{A}}{}_K(p') \underset{*}{=} \delta^{\widehat{A}}{}_K, \tag{10.9}$$

since $w^K(p')$ is an arbitrary 3-vector at p'. From (10.7) and (10.9) we get then

$$e_i{}^\alpha(p') \underset{*}{=} \delta^\alpha{}_i, \qquad e^i{}_\alpha(p') \underset{*}{=} \delta^i{}_\alpha. \tag{10.10}$$

This implies that $e_i{}^\alpha(p')$ is constant along G, and so

$$e^i{}_{\alpha,\beta}e_{\widehat{0}}{}^\beta\big|_G = \frac{\partial}{\partial s}e^i{}_\alpha(p') \underset{*}{=} e^i{}_{\alpha,0}\big|_G = 0, \qquad e^i{}_{0,\alpha}\big|_G = 0. \tag{10.11}$$

(The second equation follows from the first one by (10.3).)

Now note that

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(w^\alpha {e^i}_\alpha\right) \equiv \frac{D}{\mathrm{d}\sigma}\left(w^\alpha {e^i}_\alpha\right) = 0, \tag{10.12}$$

because w^α is tangent to a geodesic on which σ is the affine parameter, while ${e^i}_\alpha$ is parallely transported along that geodesic. Moreover, (10.8) implies $\mathrm{d}w^\alpha/\mathrm{d}\sigma = 0$, hence

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(w^\alpha {e^i}_\alpha\right) = w^\alpha \frac{\mathrm{d}{e^i}_\alpha}{\mathrm{d}\sigma} = w^\alpha w^\beta {e^i}_{\alpha,\beta} = w^A w^B {e^i}_{A,B}, \tag{10.13}$$

which means ${e^i}_{(A,B)}\big|_G = 0$, because along G w^A is an arbitrary vector orthogonal to G. Then, from (10.13)

$${e^i}_{A,B}\big|_G = 0. \tag{10.14}$$

Eqs. (10.11) and (10.14) mean that ${e^i}_{\alpha,\beta}|_G = 0$, and so, from (10.2):

$$\left.\begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}\right|_G \underset{*}{=} 0. \qquad \square \tag{10.15}$$

The physical meaning of (10.15) is that in a sufficiently small neighbourhood of each $p' \in G$, other geodesics emanating from p' are, up to first-order terms, approximated by straight lines. This neighbourhood is thus a local inertial frame. This is one more suggestion that geodesics should be the trajectories of free motion in relativity.

10.4 Special relativity vs. gravitation theory.

We have so far dealt with the postulate that general relativity should reduce to the Newtonian kinematics of free motion in the limit of vanishing gravitational field. However, in between these extremes there is special relativity that describes the kinematics of free particles in the absence of gravitation, but takes into account velocities comparable to the velocity of light. In the previous sections we have thus discussed a two-stage limiting transition: with the gravitational field to zero and with the velocity of light to infinity. When we switch off the gravitational field, but put no limits on the velocities, the geometric theory of gravitation should reproduce special relativity.

The spacetime of special relativity is a 4-dimensional flat Riemann space of signature $(+ - - -)$. The Riemann space of general relativity should thus be of the same signature. This is because a change of signature means either a discontinuity in some metric components or at least one component of the metric passing through zero value, while we expect that the "switching off" of the gravitational field can be done in a continuous way and does not lead through any singularities.

10.5 The Newtonian limit of relativity.

We assumed that with the gravitational field switched off, general relativity should reproduce special relativity, and, subsequently, with the velocity of light becoming infinite it should reproduce the Newtonian kinematics of free motion. A logical consequence of these two postulates is the requirement that when the velocity of light is made infinite while the gravitational field is still there, general relativity should reproduce the Newtonian theory of gravitation. Hence, the field equations of general relativity should, in the limit $c \to \infty$, reproduce the Poisson equation

$$\Delta\phi = 4\pi G\rho, \tag{10.16}$$

where ϕ is the gravitational potential and ρ is the density distribution of the matter generating the gravitational field.

10.6 Sources of the gravitational field.

In the theory of gravitation that we are now constructing, the gravitational field should manifest its presence as non-flatness of the metric. Consequently, the metric tensor should be the device to describe gravitation. The metric tensor in 4 dimensions has in the most general case 10 components. Hence, we will need 10 equations to determine it.

The description of the source of gravitation should thus be correspondingly elaborate, and it should be a generalisation of mass-density. According to special relativity, the mass of a body depends on its energy. Hence, the energy of motion of a continuous medium should contribute to the gravitational field, and so should the internal energy, e.g pressure in a fluid and stresses in a solid.

In special relativity, the physical state of a continuous medium is described by the energy-momentum tensor $T_{\alpha\beta}$. In any chosen coordinate system, its component T_{00} is equal to the energy-density ρc^2, the components T_{0I}, $I = 1, 2, 3$ form the 3-dimensional vector of energy stream through a unit area of surface orthogonal to the direction of flow, and the components T_{IJ} form the stress tensor. The tensor $T_{\alpha\beta}$ is symmetric, and so has in general 10 independent components. Consequently, it is a natural candidate for the source in the field equations of gravitation.

The equations of motion are differential relations between various components of the energy-momentum tensor. In the Cartesian coordinates, they are ${T^{\alpha\beta}}_{,\beta} \underset{*}{=} 0$. In any other coordinates then, they take the form

$$T^{\alpha\beta}{}_{;\beta} = 0. \tag{10.17}$$

If the $T_{\alpha\beta}$ should be the source in the field equations determining $g_{\alpha\beta}$, then the equations must be consistent with (10.17).

10.7 The Einstein equations.

Since the gravitational field should be a consequence of the geometry being non-flat, it should be connected with the curvature tensor which contains second derivatives of the metric. If the curvature is contained in the left-hand side of the field equations, then these equations will be of second order in the metric. This suggests that the components of the metric should be analogues of the Newtonian gravitational potential. In that case, the trajectories of motion (the geodesics) would be determined by the first derivatives of the potential, just like in Newton's theory.

The Riemann tensor is of rank 4, while the energy-momentum tensor is of rank 2. Hence, if the Riemann tensor were equal to some quantity constructed out of the energy-momentum tensor, then the source of the gravitational field would be a quadratic function of matter density, and this would make the transition to Newton's theory complicated. We should thus rather equate a certain quantity constructed from the Riemann tensor to the energy-momentum tensor. One good candidate is the **Ricci tensor**:

$$R_{\alpha\beta} = R^{\rho}{}_{\alpha\rho\beta}. \tag{10.18}$$

It is symmetric and linear in the second derivatives of the metric. The field equations might thus read $R_{\alpha\beta} = \kappa T_{\alpha\beta}$, where κ is a constant coefficient. However, such equations are not consistent with (10.17) because in general $R^{\alpha\beta}{}_{;\beta} \neq 0$. But the Ricci tensor obeys another identity in consequence of the Bianchi identities (7.15), which may be written as

$$R^{\alpha}{}_{\beta\gamma\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\gamma} + R^{\alpha}{}_{\beta\epsilon\gamma;\delta} = 0. \tag{10.19}$$

Contracting this equation with $\delta_{\alpha}{}^{\gamma} g^{\beta\delta}$ we get

$$G^{\alpha\beta}{}_{;\beta} = 0, \tag{10.20}$$

where

$$G^{\alpha\beta} \stackrel{\text{def}}{=} R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \qquad R \stackrel{\text{def}}{=} R^{\rho}{}_{\rho}, \tag{10.21}$$

is called the **Einstein tensor**. It is symmetric, linear in second derivatives of the metric, and obeys (10.20), which is identical in form to (10.17). Hence, $G^{\alpha\beta}$ is a better candidate for the left-hand side of the field equations, which should thus read

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa T_{\alpha\beta}. \tag{10.22}$$

These are the **Einstein equations**. The coefficient κ will be determined in Sec. 10.9 from the condition of correspondence of (10.22) to the Poisson equation (10.16).

In vacuum the Einstein equations take the form $G_{\alpha\beta} = 0$, which is equivalent to $R_{\alpha\beta} = 0$. Thus, the Ricci tensor represents that part of curvature which is algebraically determined by matter and vanishes in vacuum. The part of the curvature that propagates out of the sources into the surrounding vacuum is the traceless part of $R^{\alpha}{}_{\beta\gamma\delta}$, called the **conformal curvature tensor** or the **Weyl tensor**, defined by

$$C^{\alpha\beta}{}_{\gamma\delta} \stackrel{\text{def}}{=} R^{\alpha\beta}{}_{\gamma\delta} + \frac{1}{n-2} \delta^{\alpha\beta\rho}_{\gamma\delta\sigma} \overline{R}^{\sigma}{}_{\rho} - \frac{1}{n(n-1)} \delta^{\alpha\beta}_{\gamma\delta} R, \tag{10.23}$$

where $\overline{R}^{\sigma}{}_{\rho} \stackrel{\text{def}}{=} R^{\sigma}{}_{\rho} - \frac{1}{n} \delta^{\sigma}{}_{\rho} R$ is the traceless part of the Ricci tensor. (Note: this definition makes sense only for $n > 2$.) Studying the properties of $C^{\alpha\beta}{}_{\gamma\delta}$ is a large and important part of the relativity theory, but it is omitted in this course for lack of time and space. We mention only one important topic. In 4 dimensions, in consequence of all the (anti)symmetries in its indices and zero trace, the Weyl tensor has at most 10 independent components. These can be uniquely mapped into components of a 3×3 symmetric and traceless complex matrix. Such matrices can be classified by the degree of the minimal equation they obey and by the number of their eigenvalues. This is called the **Petrov classification** [49]. It serves as an invariant criterion for deciding whether two given metrics are genuinely inequivalent, or are just different coordinate representations of the same metric: if their Petrov types are different, then they cannot be transformed one into the other by coordinate transformations. (If their Petrov types are the same, then the question remains open.) This **equivalence problem** is until now not solved in general, and it is not even algorithmic, but attempts to solve it "in practice" are under way ([33], Chapter 9 and [50]).

It may seem unbelievable that such important and nontrivial equations should be derived almost without calculations and without experimental hints, by a speculation that was in places non-unique. However, this is how it was; Einstein actually guessed his equations by a reasoning described above in great abbreviation (it had taken him about 10 years to arrive at them.) Einstein's reasoning is described in Ref. [5], and also in Einstein's papers reprinted in Ref. [6].

Because of non-uniqueness of the reasoning leading to (10.22), Einstein's is not the only geometric theory of gravitation that can be built upon the Newton theory and special relativity. Almost every one of the intuitive assumptions made along the way can be modified so that another set of equations will result. Some of those alternative theories are briefly presented in Ref. [11]. However, Einstein's theory has successfully passed all experimental tests, whereas the competing theories were either proved inconsistent with experiment or else found to differ in their predictions from Einstein's relativity by so little that experiments cannot distinguish between them.

10.8 Hilbert's derivation of the Einstein equations.

The derivation of (10.22) in the previous section was based on Einstein's original reasoning. However, David Hilbert had worked on deriving these equations simultaneously with Einstein by a different method, and, taking things formally, was the first to publish the correct result[7] [5]. Hilbert proposed that all theories in mathematics and physics should be

[49] A. Z. Petrov, *Uchenye Zapiski Kazanskogo Gosudarstvennogo Universiteta im. V. I. Ulyanovicha-Lenina* **114**(8), 55 (1954); reprinted in English in *Gen. Relativ. Gravit.* **32**, no 8, 1665 (2000), with an editorial note by M. A. H. MacCallum, *Gen. Relativ. Gravit.* **32**, 1661 (2000) and author's biography by A. Gusev, *Gen. Relativ. Gravit.* **32**, 1663 (2000).

[50] K. Lake, *General Relativity Database.* (An interactive program to identify manifolds with given properties.) Available at www.grdb.org.

[7] Hilbert presented the subclass of (10.22) corresponding to $T_{\alpha\beta} = 0$ and to $T_{\alpha\beta}$ being the energy-momentum tensor of the electromagnetic field at the meeting of the Royal Academy of Sciences at Göttingen on 20th November 1915. Einstein presented the final correct form of (10.22) at the meeting of the Prussian Academy of Sciences in Berlin on 25th November 1915. This game with dates is only a historical curiosity. Einstein was the unquestionable spiritual father of relativity, and Hilbert himself made that point repeatedly. Einstein had worked on relativity since about 1907, and had published several papers explaining the basic ideas and preliminary results. Hilbert joined in about 1913, and was influenced by Einstein's ideas from the beginning.

derived by deduction from sets of axioms. For the gravitation theory, he postulated to use a variational principle. The reasoning leading to the action functional, just like Einstein's own reasoning, contains a few assumptions that are justified only intuitively. (Most of the other geometric theories of gravitation were derived from variational principles, which confirms that the method is not unique.)

Hilbert proposed the following axioms

I. The field equations of gravitation should follow from a variational principle. The independent variables in the action integral should be the components of the metric tensor.

II. The action functional should be a scalar.

III. The Euler–Lagrange equations (i.e. the field equations) that will follow from that functional should be differential equations of second order in $g_{\mu\nu}$.

For details of Hilbert's derivation see Ref. [11]. His action functional is $\int\limits_{\mathcal{V}} \sqrt{-g} R \mathrm{d}_4 x$, where $g = \det ||g_{\alpha\beta}||$, $R = g^{\alpha\beta} R_{\alpha\beta}$ – the trace of the Ricci tensor, and $\mathcal{V}$ is an unspecified region of the spacetime. The expression $\sqrt{-g}\mathrm{d}_4 x$ is the element of 4-dimensional volume of $\mathcal{V}$. The variations $\delta g_{\alpha\beta}$ are assumed to vanish on the boundary of $\mathcal{V}$.

Hilbert's variational principle works safely only in deriving the Einstein equations in full generality. With a less-than-general metric, for example with symmetries, the Euler – Lagrange equations for $\sqrt{-g}R$ may have nothing to do with Einstein's. This happens for a large subset of the Bianchi-type models because of their spatial homogeneity [51].

10.9 The Newtonian limit of Einstein's equations.

In Newton's theory, far from the sources the gravitational field becomes weak and should disappear in the limit of infinite distance. In that limit, the metric tensor should tend to the flat metric of special relativity. This is only a 'thought experiment' because, as mentioned earlier, the intensity of the gravitational field can never be smaller than the value determined by the average matter density in the real Universe. The spacetime whose metric becomes flat at an infinite distance from the source of the gravitational field is called **asymptotically flat**. Only in such spacetimes can we consider the Newtonian limit of general relativity. The coordinates that, in the same limit, go over into the Cartesian coordinates of special relativity are called **asymptotically Cartesian**.

From Sec. 10.5 it follows that in the limit $c \to \infty$, general relativity should reduce to Newton's theory of gravitation. Hence, in the same limit, the equation of a geodesic should reduce to the Newtonian equation of motion of a particle in a gravitational field

$$m \frac{\mathrm{d}v_I}{\mathrm{d}t} = -m\phi,_I \,, \tag{10.24}$$

where v_I is the velocity of the particle and ϕ is the gravitational potential.

Newton's equations of motion follow from the variational principle

[51] M. A. H. Mac Callum, in *General Relativity, an Einstein Centenary Survey*, Edited by S. W. Hawking and W. Israel, Cambridge University Press 1979, p. 552–553.

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = 0, \qquad \text{where} \qquad L \stackrel{\text{def}}{=} \frac{1}{2} m \delta_{IJ} v^I v^J - m\phi, \tag{10.25}$$

while the equations of a geodesic follow from another variational principle

$$\delta \int_{t_1}^{t_2} \mathcal{L} \mathrm{d}t = 0, \qquad \text{where} \qquad \mathcal{L} = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{10.26}$$

The Euler–Lagrange equations do not change when the Lagrangian is multiplied by a constant or when a constant is added to it. Hence, we require that

$$\lim_{c \to \infty} \left(C_1 \mathcal{L} + C_2 \right) = L, \tag{10.27}$$

where C_1 and C_2 are constants, as yet unknown. The Newtonian lagrangian has the dimension of energy. The relativistic lagrangian is

$$\mathcal{L} = \frac{\mathrm{d}s}{\mathrm{d}t} = c\sqrt{g_{00} + 2g_{0I}\frac{v^I}{c} + g_{IJ}\frac{v^I v^J}{c^2}}, \tag{10.28}$$

where $I, J = 1, 2, 3$ and $v^I = \mathrm{d}x^I/\mathrm{d}t$. The expression under the square root is dimensionless, so the dimension of $\mathcal{L}$ is c. The dimensions of both sides in (10.27) will thus be the same when $C_1 = \alpha mc$, where α is dimensionless.

The Newtonian L contains the kinetic energy of the particle. In special relativity, the kinetic energy is an inseparable part of the total energy $mc^2/\sqrt{1-(v/c)^2}$. Hence, C_2 in (10.27) must cancel the rest energy contained in $C_1\mathcal{L}$. We thus take $C_2 = \beta mc^2, \beta = \pm 1$, and the sign of β will follow later. Finally, (10.27) becomes

$$\lim_{c \to \infty} \left[mc^2 \left(\alpha \sqrt{g_{00} + 2g_{0I}\frac{v^I}{c} + g_{IJ}\frac{v^I v^J}{c^2}} + \beta \right) \right] = \frac{1}{2} \delta_{IJ} m v^I v^J - m\phi. \tag{10.29}$$

Developing the square root by the Taylor formula up to terms of order $(v/c)^2$ we get

$$\begin{aligned} &\sqrt{g_{00} + 2g_{0I}\frac{v^I}{c} + g_{IJ}\frac{v^I v^J}{c^2}} \\ &= \sqrt{g_{00}} \quad + \frac{g_{0I}}{\sqrt{g_{00}}} \frac{v^I}{c} + \frac{1}{2}\left(\frac{g_{IJ}}{\sqrt{g_{00}}} - \frac{g_{0I} g_{0J}}{{g_{00}}^{3/2}} \right) \frac{v^I v^J}{c^2} + O\left(\frac{v^3}{c^3} \right). \end{aligned} \tag{10.30}$$

No terms linear in v^I/c are present in L, so $c g_{0I}/\sqrt{g_{00}} \underset{c \to \infty}{\longrightarrow} 0$ must hold, i.e.

$$\frac{g_{0I}}{\sqrt{g_{00}}} = O\left(\frac{v^2}{c^2} \right). \tag{10.31}$$

Then the term $\left(-g_{0I} g_{0J}/{g_{00}}^{3/2} \right)$ in (10.30) becomes a correction of order v^4/c^4 to $g_{IJ}/\sqrt{g_{00}}$. Hence, using (10.30) and (10.31) we get in (10.29)

$$\lim_{c \to \infty} \left[c^2 \left(\alpha\sqrt{g_{00}} + \beta \right) + \alpha \, \frac{\widetilde{g}_{IJ} v^I v^J}{2\sqrt{g_{00}}} + O\left(\frac{v}{c} \right) \right] = \frac{1}{2} \, \delta_{IJ} v^I v^J - \phi, \tag{10.32}$$

where $\widetilde{g}_{IJ} = g_{IJ} - O\left(v^4/c^4\right)$. The equation above should be an identity in v^I, so

$$\begin{aligned} c^2\left(\alpha\sqrt{g_{00}} + \beta\right) &= -\phi + O\left(\frac{v}{c}\right), & (10.33)\\ \frac{\alpha}{\sqrt{g_{00}}}\,\widetilde{g}_{IJ} &= \delta_{IJ} + O\left(\frac{v}{c}\right). & (10.34)\end{aligned}$$

From (10.33) we have

$$\sqrt{g_{00}} = \frac{1}{\alpha}\left[-\beta - \frac{\phi}{c^2} + O\left(\frac{v^3}{c^3}\right)\right]. \tag{10.35}$$

We have not yet considered the condition that in the limit of vanishing gravitation, the metric in (10.26) should go over into the Minkowski metric. The term $O\left(v^3/c^3\right)$ is of higher order than ϕ/c^2, so if $\phi/c^2 \underset{c\to\infty}{\longrightarrow} 0$, then

$$\frac{1}{\alpha}\left[-\frac{\phi}{c^2} + O\left(\frac{v^3}{c^3}\right)\right] \underset{\phi\to 0}{\longrightarrow} 0. \tag{10.36}$$

Moreover, $-\beta/\alpha \underset{\phi\to 0}{\longrightarrow} 1$ must hold. But α and β are independent of c and ϕ, so $-\beta/\alpha = 1$. With the signature $(+ - --)$ (10.34) implies that $\alpha < 0$, while $\beta = \pm 1$ by definition, so finally $\alpha = -1, \beta = +1$. The final result is thus, from (10.35), (10.31) and (10.34):

$$g_{00} = 1 + \frac{2\phi}{c^2} + O\left(\frac{v^3}{c^3}\right), \qquad g_{0I} = O\left(\frac{v^2}{c^2}\right), \qquad g_{IJ} = -\delta_{IJ} + O\left(\frac{v}{c}\right). \tag{10.37}$$

Equations (10.37) follow from conditions imposed on the equations of motion. Now we shall investigate the limit $c \to \infty$ for the field equations.

From the equation $g_{\alpha\rho}g^{\beta\rho} = \delta^\beta{}_\alpha$, taking consecutively the cases $\{\alpha = 0, \beta = I \neq 0\}$, $\{\alpha = \beta = 0\}$ and $\{\alpha = I \neq 0, \beta = J \neq 0\}$ we conclude that

$$g^{00} = 1 - \frac{2\phi}{c^2} + O\left(\frac{v^3}{c^3}\right), \qquad g^{0I} = O\left(\frac{v^2}{c^2}\right), \qquad g^{IJ} = -\delta^{IJ} + O\left(\frac{v}{c}\right). \tag{10.38}$$

Using these in the formulae for the Christoffel symbols we get

$$\begin{aligned} \begin{Bmatrix} 0 \\ 00 \end{Bmatrix} &= O\left(\frac{v^3}{c^3}\right), & (10.39)\\ \begin{Bmatrix} 0 \\ 0I \end{Bmatrix} &= \frac{\phi_{,I}}{c^2} + O\left(\frac{v^3}{c^3}\right) = \begin{Bmatrix} I \\ 00 \end{Bmatrix}, & (10.40)\\ \begin{Bmatrix} 0 \\ IJ \end{Bmatrix} &= O\left(\frac{v^2}{c^2}\right) = \begin{Bmatrix} I \\ 0J \end{Bmatrix}, & (10.41)\\ \begin{Bmatrix} I \\ JK \end{Bmatrix} &= O\left(\frac{v}{c}\right). & (10.42)\end{aligned}$$

Hence,

$$R_{00} = \frac{\phi_{,II}}{c^2} + O\left(\frac{v^3}{c^3}\right) \qquad \text{(sum over } I\text{)}, \tag{10.43}$$

$$R_{0I} = O\left(\frac{v^2}{c^2}\right), \qquad R_{IJ} = O\left(\frac{v}{c}\right). \tag{10.44}$$

The Einstein equations (10.22) can be written in the equivalent form

$$R_{\alpha\beta} = \kappa\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right), \qquad \text{where} \qquad T \stackrel{\text{def}}{=} g^{\alpha\beta}T_{\alpha\beta}. \tag{10.45}$$

Since T_{00} is the energy-density, it contains a contribution from the rest energy. All other contributions to T_{00} must be by at least one order in c smaller than that, so $T_{00} = \rho c^2 + O(c)$, where ρ is the mass-density. The components T_{0I} form the vector of energy stream. In the limit $c \to \infty$, motion of matter should have no influence on the gravitational field it generates.[8] Consequently, T_{0I} should be negligible in the limit $c \to \infty$, thus $T_{0I} = O(c)$. The components T_{IJ} describe the stress energy-density. Compared to rest energy, any other kind of energy is by 2 orders in c smaller, so $T_{IJ} = O(1)$. Hence, finally

$$T = g^{\alpha\beta}T_{\alpha\beta} = \rho c^2 + O(c). \tag{10.46}$$

From here

$$R_{00} = \kappa\left(\frac{1}{2}\rho c^2 + O(c)\right), \tag{10.47}$$

$$R_{0I} = \kappa O(c), \qquad R_{IJ} = \kappa O\left(c^2\right). \tag{10.48}$$

From (10.47) and (10.43) we have

$$\frac{\phi,_{II}}{c^2} + O\left(\frac{v^3}{c^3}\right) = \kappa\left(\frac{1}{2}\rho c^2 + O(c)\right). \tag{10.49}$$

In the limit $c \to \infty$ this should be equivalent to the Poisson equation $\phi,_{II} = 4\pi G\rho$, so

$$\kappa = \frac{8\pi G}{c^4}. \tag{10.50}$$

The remaining equations in the set, (10.48), impose limitations on the terms $O(v^2/c^2)$ in g_{0I} and $O(v/c)$ in g_{IJ}. These can be read out if we want to consider the Einstein theory as a small perturbation imposed on the Newton theory in a weak gravitational field. There are various approaches to this application of general relativity, the most modern of them is the Parametrized Post-Newtonian (PPN) formalism, see Refs. [52] and [4]. Formally, in the limit $c \to \infty$ eqs. (10.48) are fulfilled identically.

In the equation $R_{00} = \kappa\left(T_{00} - \frac{1}{2}g_{00}T\right)$ we had to include terms of order $1/c^2$ because the relativistic time coordinate is $x^0 = ct$, and so every differentiation by x^0 introduces the factor $1/c$. In order to liberate R_{00} from the factor $1/c^2$ introduced in this way, we had to multiply (10.49) by c^2. To achieve the same in Eqs. (10.48) we would have to multiply them by c, and then they would turn into the identities $0 = 0$ in the limit $c \to \infty$.

---

[8] If the mass distribution does not change with time, then its motion is not detectable via its gravitational field. For example, the exterior gravitational field of mass in axisymmetric rotation in Newtonian theory is not distinguishable from the field generated by a static mass with the same distribution.

[52] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*. Freeman, San Francisco 1973.

10.10 Examples of sources in the Einstein equations: perfect fluid and dust.

A fluid whose pressure is isotropic while the transport of energy occurs only by means of mass-flow is called a **perfect fluid**. It does not conduct heat or electric current, and its viscosity is zero. We will now deduce the form of its energy-momentum tensor.

If u^α is the velocity field of an arbitrary continuous medium, while s is the proper time on the lines of flow of that medium, then

$$\mathrm{d}s^2 = g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \qquad \text{and} \qquad u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}s}. \tag{10.51}$$

These imply

$$g_{\alpha\beta}u^\alpha u^\beta = 1. \tag{10.52}$$

This holds *for every continuous velocity field* and for any medium.

Now let us choose, for a while, the coordinates adapted to u^α, in which $u^\alpha \underset{*}{=} \delta^\alpha{}_0$, i.e. the time-coordinate x^0 is the affine parameter on the flow lines of matter, and $\mathrm{d}x^I(s)/\mathrm{d}s \underset{*}{=} 0, I = 1,2,3$, where $x^I(s)$ are spatial coordinates of the particles of the fluid. Thus, $x^I = \text{const}$, which means that the particles of the fluid do not move with respect to the timelike hypersurfaces of the coordinate system. Such coordinates are called **comoving**. Their construction can be visualised as follows: We choose an arbitrary hypersurface S that intersects all flow lines of the fluid, and an arbitrary coordinate system within S. To each particle of the fluid we then assign the spatial coordinate x^I of the point where it crossed S. To assign the time-coordinate to a point p in spacetime we take the flow line C_p that passes through p. The time-coordinate of p is the proper time that elapsed between the event of C_p intersecting S and the event of C_p passing through p.

By definition, $T_{00}(p)$ is the energy density measured at a given point p. In comoving coordinates, the only energy that a particle can have is its inner energy ϵ = (rest energy) + (energy of thermal motion of its particles) + (chemical energy). Hence, $T_{00} \underset{*}{=} \epsilon$, and, at the same time, $T_{00} \underset{*}{=} T_{\alpha\beta}u^\alpha u^\beta$, so

$$T_{\alpha\beta}u^\alpha u^\beta = \epsilon. \tag{10.53}$$

This is now an equality of two scalars, so it holds in any coordinates.

Again by definition, $T^I{}_0$, $I = 1,2,3$ is the vector of the energy stream. But in the comoving coordinates, there are no energy flows in a perfect fluid, so $T^I{}_0 \underset{*}{=} T^I{}_\beta u^\beta \underset{*}{=} 0$. This implies that $T^\alpha{}_\beta u^\beta = \lambda u^\alpha$, and from (10.52) and (10.53) we find that $\lambda = \epsilon$, so

$$T^\alpha{}_\beta u^\beta = \epsilon u^\alpha, \tag{10.54}$$

which is again a tensor equation.

Now choose a point q and an arbitrary vector $v^\alpha(q)$ that is orthogonal to $u^\alpha(q)$:

$$u^\alpha(q)v_\alpha(q) = 0. \tag{10.55}$$

The vector $v^\alpha(q)$ points from q toward a neighbouring particle. Since eq. (10.55) says that the projection of the velocity on $v^\alpha(q)$ is zero, it follows that the particle to which $v^\alpha(q)$ points does not move relative to q. The collection of all vectors having the property (10.55) thus determines a 3-dimensional volume element comoving with the particle that was at q. Consequently, the Pascal law must apply in this volume element: the pressure p exerted on the surface element σ in the fluid creates the force $f = p\sigma$ in the direction n^I, $I = 1, 2, 3$ orthogonal to σ. The pressure p and the force f do not depend on the direction of n^I, i.e. on the orientation of σ within the fluid. Let $-T^I{}_J$ denote the Newtonian (3-dimensional) stress tensor. (The minus sign is a consequence of the signature $(+ - --)$, in which the spatial part of the energy-momentum tensor $T^\alpha{}_\beta$ is not the stress tensor $\tau^I{}_J$ itself, but $-\tau^I{}_J$.) By the definition of the stress tensor, the following must hold

$$-T^I{}_J \sigma n^J = f n^I \equiv p\sigma n^I, \tag{10.56}$$

which implies

$$T^I{}_J n^J = -p n^I. \tag{10.57}$$

The vector n^I is an arbitrary vector in the 3-dimensional subspace orthogonal to $u^\alpha(q)$. The equation (10.57) shows that every such vector is an eigenvector of the matrix $T^I{}_J$ connected with the eigenvalue $(-p)$, which implies

$$T^I{}_J = -p\delta^I{}_J. \tag{10.58}$$

The vectors n^I in (10.56) – (10.58) are in general not orthogonal to any hypersurfaces, and so cannot define a coordinate system. However, at every point q of the manifold, in the tangent space we can choose an orthonormal basis $e_{\widehat{I}}{}^\alpha$, $I = 1, 2, 3$ orthogonal to u^α. In that basis, eq. (10.58) will be fulfilled, where

$$T_{\widehat{I}\widehat{J}} = e_{\widehat{I}}{}^\alpha e_{\widehat{J}}{}^\beta T_{\alpha\beta}. \tag{10.59}$$

(In other words, we choose coordinates that are adapted to the basis vectors $e_{\widehat{I}}{}^\alpha$ at one point q only, which is always possible.)

Now, from (10.53) – (10.54), (10.58) and (10.59) the formula for the energy-momentum tensor of a perfect fluid will follow. Let us choose an orthonormal tetrad $e_i{}^\alpha$, $i = 0, 1, 2, 3$ in spacetime such that $e_{\widehat{0}}{}^\alpha = u^\alpha$, while each $e_{\widehat{I}}{}^\alpha$, $I = 1, 2, 3$ obeys (10.57). Then

$$g_{\alpha\beta} e_i{}^\alpha e_j{}^\beta = \eta_{ij} = \mathrm{diag}(+1, -1, -1, -1) \tag{10.60}$$

and from (10.53) – (10.54), (10.59) and (10.60) we have

$$T_{\widehat{0}\widehat{0}} = T_{\alpha\beta} u^\alpha u^\beta = \epsilon, \tag{10.61}$$

$$T_{\widehat{0}\widehat{A}} = T_{\alpha\beta} u^\alpha e_{\widehat{A}}{}^\beta = \epsilon u_\beta e_{\widehat{A}}{}^\beta = 0, \qquad A = 1, 2, 3, \tag{10.62}$$

$$T_{\widehat{A}\widehat{B}} = -T^{\widehat{A}}{}_{\widehat{B}} = p\delta^{\widehat{A}}{}_{\widehat{B}} = -p\eta_{\widehat{A}\widehat{B}}. \tag{10.63}$$

Applying the inverse projection, (4.16), we get

$$\begin{aligned} T_{\alpha\beta} &= e^i{}_\alpha e^j{}_\beta T_{ij} = u_\alpha u_\beta T_{\widehat{0}\widehat{0}} + e^{\widehat{A}}{}_\alpha e^{\widehat{B}}{}_\beta T_{\widehat{A}\widehat{B}} \\ &= \epsilon u_\alpha u_\beta - p\eta_{\widehat{A}\widehat{B}} e^{\widehat{A}}{}_\alpha e^{\widehat{B}}{}_\beta - p u_\alpha u_\beta + p u_\alpha u_\beta \\ &= (\epsilon + p) u_\alpha u_\beta - p g_{\alpha\beta}. \end{aligned} \tag{10.64}$$

A perfect fluid whose pressure is identically zero is called **dust**. It follows that for dust

$$T_{\alpha\beta} = \epsilon u_\alpha u_\beta. \tag{10.65}$$

10.11 Equations of motion of a perfect fluid.

The equations of motion of the sources of a gravitational field, $T^{\alpha\beta};_\beta = 0$, for a general perfect fluid are equivalent to

$$(\epsilon + p),_\beta u^\alpha u^\beta + (\epsilon + p)u^\alpha;_\beta u^\beta + (\epsilon + p)u^\alpha u^\beta;_\beta - p,_\beta g^{\alpha\beta} = 0. \tag{10.66}$$

The identity (10.52) implies

$$u^\alpha u_{\alpha;\beta} = 0. \tag{10.67}$$

Contracting (10.66) with u_α and using (10.52) and (10.67) we obtain

$$\epsilon,_\beta u^\beta + (\epsilon + p)u^\beta;_\beta = 0. \tag{10.68}$$

This is the energy conservation equation which says that the volume work $-pu^\beta;_\beta$ generates the energy stream ϵu^β. Now using (10.68) in (10.66) we get

$$(\epsilon + p)u^\alpha;_\beta u^\beta - p,_\beta g^{\alpha\beta} + p,_\beta u^\beta u^\alpha = 0. \tag{10.69}$$

These are the general relativistic equations of motion of a perfect fluid. In the Newtonian limit ($c \to \infty$) and in asymptotically Cartesian coordinates, they go over into the Euler equations of motion of a perfect fluid $\rho \mathrm{d}\mathbf{v}/\mathrm{d}t = -\mathrm{grad}p$.

Eqs. (10.68) and (10.69) simplify for dust, when $p = 0$. Equation (10.68) then becomes

$$\left(\epsilon u^\beta\right);_\beta = 0, \tag{10.70}$$

which is the relativistic equation of continuity (mass conservation) that in the Newtonian limit goes over into $\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0$. Equation (10.69) becomes

$$u^\alpha;_\beta u^\beta = 0. \tag{10.71}$$

This one means that the covariant derivative of the vector field tangent to the flow lines along these lines is zero. This fulfils the requirements of the definition of a geodesic in affine parametrisation, see Sec. 5.2. Consequently, *dust moves along geodesic lines*, and *the proper time of the particles of dust is an affine parameter on these geodesics.*

The necessary and sufficient condition for a geodesic motion of a perfect fluid is weaker than $p = 0$. As seen from (10.69), it is $p,_\beta \left(g^{\alpha\beta} - u^\alpha u^\beta\right) = 0$. The operator $h^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ projects tensors on the hypersurface elements orthogonal to the vector field u^α. The equation $h^{\alpha\beta} p,_\alpha = 0$ means that the gradient of pressure is collinear with the velocity field, i.e. there are no spatial gradients of pressure in the comoving coordinates.

10.12 The cosmological constant.

It was clear from the beginning that the predictions of general relativity would significantly differ from those of Newton's theory in two situations:

1. In strong gravitational fields, when the region where local inertial frames exist becomes small.

2. In large sub-volumes of the Universe, where even small departures of the real geometry from the Euclidean geometry cumulate over large distances and become visible during observations of distant objects.

Consequently, when Einstein thought about physical applications of his newly invented theory, one of the first things he tried was to construct a model of the Universe. At that time everybody was *sure* that the geometry and matter distribution in the Universe are time- and space-independent. Calculations showed, however, that under these assumptions the Einstein equations allow no solutions.

At that moment, Einstein was on the verge of making another great discovery. The year was 1916, while the first observations proving the expansion of the Universe were published only in 1927. Yet this time Einstein's belief in prejudice turned out to be stronger than his tendency to think boldly. While searching for a reason of the failure of his first attempt, Einstein did not question the *assumption* that the Universe is static, but turned his suspicion against his equations (10.22). He had soon found a gap in his reasoning: the left-hand side of the field equations that has all the required properties need not necessarily be the Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. One can add to it any symmetric tensor $H_{\mu\nu}$ whose covariant divergence vanishes and which does not depend on the derivatives of the metric. Such a correction will not increase the order of the field equations, while the equation $\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + H^{\mu\nu}\right)_{;\nu} = 0$ will still hold. The simplest tensor of this property is $\Lambda g^{\mu\nu}$, where Λ should be a universal constant. The modified Einstein equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \tag{10.72}$$

in the limit $c \to \infty$ go over into the modified Poisson equation

$$\Delta\phi - \Lambda c^2 = 4\pi G\rho. \tag{10.73}$$

Hence, the constant Λ, called the **cosmological constant**, describes an effect that is absent from the ordinary Newton theory: a universal attraction (for $\Lambda > 0$) or repulsion (for $\Lambda < 0$) of matter particles.

For the modified equations (10.72) the following static, homogeneous and isotropic solution exists:

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - R^2 \mathrm{d}\chi^2 - R^2 \sin^2\chi \left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right), \tag{10.74}$$

where

$$R = \frac{1}{\sqrt{-\Lambda}} = \frac{c}{2\sqrt{\pi G\rho}}\ , \tag{10.75}$$

ρ being the average mass density in the Universe, constant by assumption. The spacetime corresponding to the metric (10.74) is, by tradition, still called the **Einstein Universe**, although it is no longer considered a model of the real Universe. Since $\Lambda < 0$ in it, the "cosmological repulsion" balances the gravitational attraction, which allows the the system to be static (but unstable against perturbations of (10.75), as we will see in Sec. 14.8).

This brief story shows that in fact the cosmological constant appeared in consequence of an error. Had Einstein not insisted on obtaining a static model of the Universe, he would have predicted that the Universe should expand or collapse, 11 years before the expansion was observed. When he realised later how close he was to making that prediction, he said that the introduction of the constant was "the biggest blunder of his life" [53].

That story has a complicated continuation. Hubble, who is credited with the discovery (today in fact *re*discovery) of the expansion of the Universe [54], had not believed, until the end of his life, that the Universe is actually expanding. He insisted, even in his last paper that appeared in print after his death [55], that expressing the observed redshifts in spectra of galaxies through their equivalent velocities of recession is merely a convenient mathematical device. See Ref. [56] for more details.

Then, in 1934, Milne and McCrea [57] showed that expansion of the Universe could and should have been predicted on the basis of Newton's theory of gravitation in the 18th century since all the mathematical knowledge necessary for that purpose had existed already at that time. The prediction was not made because nobody tried – just because everybody was sure the Universe was static.

Since about 2000, the cosmological constant is back in favour. A Λ-like medium called "dark energy" is believed to propel the expansion of the Universe at an accelerated rate. This is supposed to explain the observations of supernovae of type Ia, whose distances were found inconsistent with a *Robertson – Walker Universe* expanding with deceleration. The concept of "dark energy" is now accepted in astronomy with an unshakable faith, and its authors even obtained a Nobel prize for their work. Still, it was shown by a few authors that the existence of dark energy is a consequence of using the Robertson – Walker class of models to interpret the observations. Using the more general Lemaître–Tolman class (see our chapter 15) one can explain the supernovae observations by irregularities in the expansion caused by an inhomogeneous distribution of matter in the Universe; see Refs. [58] and [59]. This alternative explanation has no implications for the value of Λ, the possibility of $\Lambda = 0$ being not excluded.

The absolute value of Λ, if nonzero, must be less than 10^{-50} cm^{-2} [53], so it can play a role only in the evolution of the Universe. In the Solar System, it has no observable influence on the motion of planets. Still, a great number of solutions of the modified Einstein equations (10.72) is known [33], both inside matter distributions and in vacuum.

10.13 Matching solutions of Einstein's equations

Just as in electrodynamics, in gravitation theory we sometimes have to match solutions of Einstein's equations obtained separately for different spacetime regions. Most often, we want to determine whether a given vacuum solution can be interpreted as the exterior gravitational field to a material body for whose interior we have found another solution. After matching, the arbitrary constants in the vacuum region are determined in terms of the parameters of the interior metric.

---

[53] Page 410 – 411 in Ref. [52].

[54] E. P. Hubble, *Proc. Nat. Acad. Sci. USA* **15**, 169 (1929).

[55] E. P. Hubble, *Mon. Not. Roy. Astr. Soc.* **113**, 658 (1953).

[56] G. F. R. Ellis and A. Krasiński, *Gen. Relativ. Gravit.* **31**, 1985 (1999).

[57] E. A. Milne, *Quart. J. Math. Oxford* **5**, 64 (1934); W. H. McCrea and E. A. Milne, *Quart. J. Math. Oxford* **5**, 73 (1934); both papers reprinted in *Gen. Relativ. Gravit.* **32**, 1939 and 1949 (2000), with an editorial note and authors' biographies by A. Krasiński, *Gen. Relativ. Gravit.* **32**, 1933 and 1935 (2000).

[58] M. N. Celerier, *Astron. Astrophys.* **353**, 63 (2000).

[59] H. Iguchi, T. Nakamura and K. Nakao, *Progr. Theor. Phys.* **108**, 809 (2002).

Sometimes we are interested in describing a surface matter distribution on the boundary between two regions, but we shall exclude this case here. Generally, we assume that there are no singularities of the type of Dirac δ function in components of the curvature tensor and that the hypersurface across which we match the two metrics is non-null. (Together with surface matter distributions we thus exclude shock waves in which the discontinuity in the Riemann tensor is matched to vacuum solutions on both sides. Null matching hypersurfaces pose additional problems.)

Under these assumptions, the components of the Riemann tensor can at worst be discontinuous across the matching hypersurface Σ. We have to allow discontinuities because, for example, the mass-density of a perfect fluid body is nonzero at its surface, but zero at adjacent vacuum points – and it is related to the Ricci tensor via (10.64) and (10.22).

In order to discuss the matching conditions, it is most convenient to use the coordinates adapted to Σ introduced at the end of Section 7.11. This time, $n = 3$, $V_n = \Sigma$, and the spacetime metric in the adapted coordinates is

$$ds^2 = g_{ab}dx^a dx^b + \varepsilon\mathcal{N}^2\left(dx^4\right)^2, \qquad a, b = 1, 2, 3, \tag{10.76}$$

where the x^4 coordinate may be timelike (then $\varepsilon = +1$) or spacelike ($\varepsilon = -1$). The unit normal vector to the boundary $x^4 = A =$ constant is $X^\alpha = (0, 0, 0, 1/\mathcal{N})$. For $x^4 > A$ we have one metric $g^+_{\alpha\beta}$ (e.g. vacuum); for $x^4 < A$ we have another $g^-_{\alpha\beta}$ (e.g. the interior of a material body). Since Σ has to be the same, whichever of the two 4-metrics is used to describe it, the components g_{ab} have to be continuous across Σ. Since on Σ they are functions of the x^a-coordinates, the same functions no matter which 4-metric is used to calculate the $g_{ab}(\Sigma)$, the continuity of all the derivatives of g_{ab} along Σ, i.e. of the derivatives by x^a, is automatically guaranteed, and we need only take extra care about $g_{ab,4}$.

So far, we have made sure that the boundary hypersurface Σ has the same intrinsic geometry in both metrics. However, if the regions on the opposite sides of Σ are to be parts of the same manifold, then Σ must have the same extrinsic geometry with respect to both metrics, i.e. be embeddable in both in the same way. Without that, we might end up trying to identify a cylinder with a plane, for example. Hence, the second fundamental form of Σ must be the same, whichever of the two metrics is used to calculate it.

Thus, finally, the conditions that two metrics, $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$, can be considered to describe two parts of the same manifold are as follows:

A hypersurface Σ must exist such that, in coordinates adapted to Σ as in (10.76), the metrics induced on Σ are the same ($g^+_{ab}(\Sigma) = g^-_{ab}(\Sigma)$), and the second fundamental form of Σ must be the same, whichever metric, $g^+_{\alpha\beta}$ or $g^-_{\alpha\beta}$, is used to calculate it.

Now we will investigate the consequences of these conditions for the Riemann and Einstein tensors of the spacetime and for the Riemann tensor of Σ using the Gauss – Codazzi equations (7.62) and (7.63). With $X^\alpha = (0, 0, 0, 1/\mathcal{N})$ and x^α coinciding with x^a in Σ for $\alpha, a = 1, 2, 3$, Eqs. (7.62) and (7.63) become

$$R_{dabc}(\Sigma) = R_{dabc}(V_4) + \varepsilon\left(\Omega_{ac}\Omega_{db} - \Omega_{ab}\Omega_{dc}\right), \tag{10.77}$$

$$\Omega_{ab;c} - \Omega_{ac;b} = -R_{4abc}(V_4)/\mathcal{N}. \tag{10.78}$$

Since $R_{dabc}(\Sigma)$ and Ω_{ab} are continuous across Σ, the Σ-components of the 4-dimensional Riemann tensor, $R_{dabc}(V_4)$, will be continuous across Σ, too. The covariant derivatives of Ω_{ab} in (10.78) are taken within Σ, so they are also continuous. This means that $R_{4abc}(V_4)/\mathcal{N}$ has to be continuous across Σ, but $\mathcal{N}$ and $R_{4abc}(V_4)$ individually need not be – they may have a discontinuity that cancels out in the quotient.

We still have to consider the components $R_{4a4b}(V_4)$ that are not determined by the Gauss – Codazzi equations. They are, using (7.65) in the square brackets

$$\begin{aligned} R_{4a4b} = \varepsilon\mathcal{N}^2 R^4{}_{a4b} = \varepsilon\mathcal{N}^2 \Bigg[-\frac{1}{4} g^{44}{}_{,b}\, g_{44,a} - \frac{1}{2} g^{44} g_{44,ab} - \frac{1}{4} g^{44}{}_{,4}\, g_{ab,4} - \frac{1}{2} g^{44} g_{ab,44} \\ + \frac{1}{2} g^{44} g_{44,c} \left\{ {c \atop ab} \right\}_{(\Sigma)} + \frac{1}{4} g^{44} g^{cd} g_{cb,4} g_{ad,4} \Bigg] \\ = \varepsilon\mathcal{N}\left(-\mathcal{N}_{;ab} + \varepsilon\mathcal{N} g^{cd}\Omega_{cb}\Omega_{ad} + \varepsilon\mathcal{N}\Omega_{ab,\rho}X^\rho \right). \end{aligned} \tag{10.79}$$

These components contain terms that are not intrinsic to Σ, like $g^{44}g_{ab,44}$, so they can be discontinuous across Σ.

Now we find for the components of the Einstein tensor:

$$\begin{aligned} G^4{}_4 &= -\frac{1}{2} g^{ab} R^c{}_{acb}, \qquad G_{4a} = R^c{}_{4ca}, \\ G_{ab} &= R^c{}_{acb} + R^4{}_{a4b} - \frac{1}{2} g_{ab} g^{cd} R^e{}_{ced} - g_{ab} g^{cd} R^4{}_{c4d}. \end{aligned} \tag{10.80}$$

This shows that $G^4{}_4$ is continuous across Σ, while G_{4a} and G_{ab} can be discontinuous.

Note what this means for matching a perfect fluid solution to vacuum across a timelike hypersurface such that the velocity has no component in the x^4 direction (i.e. is tangent to Σ). Then $G^4{}_4 = \kappa p$, and since $G^4{}_4 = 0$ in vacuum and has to be continuous across Σ, this means that $p = 0$ must hold on Σ.

10.14 Exercises.

1. Verify that the Newtonian limit of eqs. (10.69) are the Euler equations of motion of a perfect fluid $\rho \mathrm{d}\mathbf{v}/\mathrm{d}t = -\mathrm{grad}p$.

Hint: Use the asymptotically Cartesian coordinates in which the metric has the form (10.37) – (10.38), and observe that in these coordinates the four-velocity u^α has the form $u^0 = 1$, $u^a = v^a/c$, $a = 1, 2, 3$.

2. Verify that eq. (10.70) becomes the continuity equation $\partial\rho/\partial t + \mathrm{div}(\rho\mathbf{v}) = 0$ in the Newtonian limit.

3. Verify that eqs. (10.74) – (10.75) indeed represent a solution of the modified Einstein equations (10.72). The energy-momentum tensor is that of a perfect fluid, the velocity field is $u^\alpha \underset{*}{=} \delta^\alpha{}_0/c$. Find the pressure.