Chapter 15

Relativistic cosmology III: the Lemaître–Tolman geometry.

15.1 The comoving-synchronous coordinates.

The curvature coordinates of Chapter 11 were convenient for investigating the Schwarzschild solution. However, for some other purposes the **comoving coordinates** are more useful. They can be introduced whenever there exists a timelike vector field u^α on the spacetime. By definition, in these coordinates the (contravariant) vector field u^α has only the u^0 component. They exist always – one has to solve the set of three equations $u^\alpha x^{\alpha'}{}_{,\alpha} \underset{*}{=} 0$ for $\alpha' = 1, 2, 3$. Note that nothing happens to the x^0-coordinate thereby. The transformations preserving them are $x^I = f^I(x^{J'})$, $x^0 =$ any function of all four coordinates.

If u^α is the velocity field of matter and has zero rotation, then the comoving coordinates can be chosen so that, in addition, they are **synchronous**, that is, the metric tensor has no time-space components. Here is the proof that $\omega = 0$ is a necessary condition:

Suppose comoving-synchronous coordinates $\{x^\alpha\}$ exist. Then $u^\alpha = \lambda^{-1}\delta_0{}^\alpha$, $u_\alpha = \lambda\delta^0{}_\alpha$, $g_{00} = \lambda^2$, $g_{0I} = g^{0I} = 0$ and $\dot{u}_\alpha = (\lambda,_0 \delta^0{}_\alpha - \lambda,_\alpha)/\lambda$, and using (12.31) we find $\omega_{\alpha\beta} = 0$. □

Now suppose $\omega_{\alpha\beta} = 0$; then $2\sqrt{-g}w^\alpha = \epsilon^{\alpha\beta\gamma\delta}u_\beta u_{\gamma,\delta} = 0$. This can be written as $u \wedge \mathrm{d}u = 0$, where $u = u_\alpha \mathrm{d}x^\alpha$. Then there exist such functions f and λ that $u = \lambda \mathrm{d}f$ ([136], Sec. 7.2). Choosing $x^{0'} = f$ we ensure that $u_{\alpha'} = \lambda\delta^0{}_{\alpha'}$. Now recall the remark above – while introducing the comoving coordinates nothing happens to $x^{0'}$, so $u_{\alpha''} = \lambda\delta^0{}_{\alpha''}$ still holds, and we achieve $u^{\alpha''} \propto \delta_0{}^{\alpha''}$ in addition. This implies that $g_{\alpha''\beta''}$ has no time-space components, i.e., the final coordinates are comoving and synchronous.

[136] H. Flanders, *Differential forms with applications to physical sciences.* Academic Press, New York and London 1963.

15.2 Spherically symmetric inhomogeneous models.

For a spherically symmetric perfect fluid, rotation is zero (see Exercise 1). Hence, the comoving-synchronous coordinates can be introduced, in which the metric becomes

$$\mathrm{d}s^2 = \mathrm{e}^{C(t,r)}\mathrm{d}t^2 - \mathrm{e}^{A(t,r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right), \tag{15.1}$$

and the velocity field is

$$u^\alpha = \mathrm{e}^{-C/2}\delta^\alpha{}_0. \tag{15.2}$$

The function R (called the **areal radius**) is the source area distance, defined in (13.55), between an observer at arbitrary (t, r) and the centre $R = 0$.

The Einstein equations for (15.1), with the cosmological constant included, are

$$\begin{aligned}
G^0{}_0 &= \mathrm{e}^{-C}\left(\frac{R_{,t}{}^2}{R^2} + \frac{A_{,t}R_{,t}}{R}\right) - \mathrm{e}^{-A}\left(2\frac{R_{,rr}}{R} + \frac{R_{,r}{}^2}{R^2} - \frac{A_{,r}R_{,r}}{R}\right) + \frac{1}{R^2}\\
&= \kappa\epsilon - \Lambda, && (15.3)\\
G^1{}_0 &= \mathrm{e}^{-A}\left(2\frac{R_{,tr}}{R} - \frac{A_{,t}R_{,r}}{R} - \frac{R_{,t}C_{,r}}{R}\right) = 0, && (15.4)\\
G^1{}_1 &= \mathrm{e}^{-C}\left(2\frac{R_{,tt}}{R} + \frac{R_{,t}{}^2}{R^2} - \frac{C_{,t}R_{,t}}{R}\right) - \mathrm{e}^{-A}\left(\frac{R_{,r}{}^2}{R^2} + \frac{C_{,r}R_{,r}}{R}\right) + \frac{1}{R^2}\\
&= -\kappa p - \Lambda, && (15.5)\\
G^2{}_2 &= G^3{}_3 = \frac{1}{4}\mathrm{e}^{-C}\left(4\frac{R_{,tt}}{R} - 2\frac{C_{,t}R_{,t}}{R} + 2\frac{A_{,t}R_{,t}}{R} + 2A_{,tt} + A_{,t}{}^2 - C_{,t}A_{,t}\right)\\
&- \frac{1}{4}\mathrm{e}^{-A}\left(4\frac{R_{,rr}}{R} + 2\frac{C_{,r}R_{,r}}{R} - 2\frac{A_{,r}R_{,r}}{R} + 2C_{,rr} + C_{,r}{}^2 - C_{,r}A_{,r}\right)\\
&= -\kappa p - \Lambda. && (15.6)
\end{aligned}$$

Equation (15.3), multiplied by $R^2R_{,r}$, may be rewritten in the equivalent form

$$\left(R + \mathrm{e}^{-C}RR_{,t}{}^2 - \mathrm{e}^{-A}RR_{,r}{}^2 + \frac{1}{3}\Lambda R^3\right)_{,r} - R\left(\mathrm{e}^{-C}R_{,t}{}^2\right)_{,r} + \mathrm{e}^{-C}A_{,t}RR_{,t}R_{,r} = \kappa\epsilon R^2R_{,r}\,. \tag{15.7}$$

But last two terms on the left-hand side sum up to zero in consequence of (15.4), so

$$\left(R + \mathrm{e}^{-C}RR_{,t}{}^2 - \mathrm{e}^{-A}RR_{,r}{}^2 + \frac{1}{3}\Lambda R^3\right)_{,r} = \kappa\epsilon R^2R_{,r}\,. \tag{15.8}$$

By a similar method, multiplying by $R^2R_{,t}$, we transform eq. (15.5) to the equivalent form

$$\left(R + \mathrm{e}^{-C}RR_{,t}{}^2 - \mathrm{e}^{-A}RR_{,r}{}^2 + \frac{1}{3}\Lambda R^3\right)_{,t} = -\kappa pR^2R_{,t}\,. \tag{15.9}$$

From (15.8) we can now recognise that the quantity

$$m \stackrel{\text{def}}{=} \frac{c^2}{2G}\left(R + \mathrm{e}^{-C}RR_{,t}{}^2 - \mathrm{e}^{-A}RR_{,r}{}^2 + \frac{1}{3}\Lambda R^3\right) \tag{15.10}$$

has the property expected from mass: assuming $R = 0$ at $r = r_0$ and integrating (15.8) from r_0 to a current r in a space of constant t, we get $m(r) = \int_{r_0}^{r} 4\pi\epsilon c^2 R^2 R,_{r'} \mathrm{d}r' \equiv \int_0^{R(r)} 4\pi\epsilon c^2 u^2 \mathrm{d}u$. At the surface of a spherical star, where $p = 0$, (15.9) reduces to $m,_t = 0$, i.e. the total mass of a star embedded in vacuum is conserved.

In deriving (15.8) – (15.9) we assumed $R,_r \neq 0 \neq R,_t$. The case $R,_t = 0$ is not interesting for cosmology: then, from (15.4), either $R,_r = 0$, which leads to the Nariai solution [62], or $A,_t = 0$, which, by (15.3), has time-independent density. But with $R,_r = 0$, eqs. (15.3) – (15.6) lead to the Datt – Ruban solution [42, 45, 46]. So far, it has found no application in astrophysics, but has interesting physical properties. [45, 46, 11, 137]

The definition (15.10) (with $\Lambda = 0$) was introduced by Lemaître [91], and then rederived by Podurets in 1964 [138], but it is most often credited to Misner and Sharp [139].

15.3 The Lemaître–Tolman model.

To solve (15.3) – (15.6), an equation of state must be assumed. A barotropic equation $\epsilon = f(p)$ does not seem to be right as it implies the entropy per particle to be a universal constant [11]. Lacking any better idea, the simplest choice is $p = 0$, which we assume now.

As seen from (10.69), with $p = 0$ perfect fluid particles move on timelike geodesics. Acceleration being equal to zero then implies $C,_r = 0$ in (15.1) (see Exercise 2), and the transformation $t' = \int \mathrm{e}^{C/2}\mathrm{d}t$ leads to $C = 0$. Then (15.4) becomes

$$\left(\mathrm{e}^{-A/2}R,_r\right),_t = 0. \tag{15.11}$$

With the case $R,_r = 0$ set aside, we assume $R,_r \neq 0$. Then the solution of (15.11) is

$$\mathrm{e}^A = \frac{{R,_r}^2}{1 + 2E(r)}, \tag{15.12}$$

where $E(r)$ is an arbitrary function. For the metric (15.1) to have the right signature it is necessary that $E \geq -1/2$ for all r.[48] Using $C = 0$ and (15.12) to eliminate ${R,_r}^2$ from (15.5) we obtain

$$2\frac{R,_{tt}}{R} + \frac{{R,_t}^2}{R^2} - \frac{2E(r)}{R^2} + \Lambda = 0. \tag{15.13}$$

Since we assumed $R,_t \neq 0$, we multiply (15.13) by $R^2 R,_t$ and then integrate, obtaining

$${R,_t}^2 = 2E(r) + \frac{2M(r)}{R} - \frac{1}{3}\Lambda R^2, \tag{15.14}$$

$M(r)$ being one more arbitrary function. With $\Lambda = 0$, (15.14) is formally identical to the Newtonian equation of radial motion in a Coulomb potential. In this Newtonian

[137] A. Krasiński and G. Giono, *Gen. Relativ. Gravit.* **44**, 239 (2012).
[138] M. A. Podurets, *Astron. Zh.* **41**, 28 (1964); English translation: *Sov. Astr. A. J.* **8**, 19 (1964).
[139] C. W. Misner and D. H. Sharp, *Phys. Rev.* **B136**, 571 (1964).

[48] The value $E = -1/2$ is admissible provided $R,_r = 0$ at the same location. This is a *neck* or *wormhole* – see Refs. [140, 11] for more on this point.

[140] C. Hellaby and K. Lake, *Astrophys. J.* **290**, 381 (1985) + erratum *Astrophys. J.* **300**, 461 (1985).

analogy, $c^2M(r)/G$ is the active gravitational mass within an r = constant shell, and $c^2E(r)/G$ is the total energy within the same shell. The solution of (15.14) will contain one more arbitrary function, $t_B(r)$ called the **bang-time function**, that will appear in the combination $(t - t_B(r))$. We will see below that when $\Lambda = 0$, $t = t_B(r)$ is indeed the time coordinate of the Big Bang (BB) singularity, and it is in general position-dependent. With $\Lambda \neq 0$, the BB will not always occur, just as in the Friedmann models.

With (15.11), (15.13) and $C = 0$ (15.6) is now an identity, while (15.3) implies

$$\frac{8\pi G}{c^4}\epsilon = \frac{2M,_r}{R^2 R,_r}. \tag{15.15}$$

The mass-density ϵ/c^2 becomes infinite where $R = 0 \neq M,_r$ and where $R,_r = 0 \neq M,_r$. The first of these is the BB, which is unavoidable when $\Lambda = 0$. The second is a shell crossing: there, the radial geodesic distance between the points $\{t_0, r_0, \vartheta_0, \varphi_0\}$ and $\{t_0, r_0{+}\mathrm{d}r, \vartheta_0, \varphi_0\}$, equal to $\sqrt{|g_{rr}|}\mathrm{d}r$, becomes zero, which means that shells of different values of r intersect. This singularity can be avoided with an appropriate choice of $M(r)$, $E(r)$ and $t_B(r)$. In the Friedmann limit (see below), the shell crossing coincides with the BB.

The final solution of Einstein's equations is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{{R,_r}^2}{1 + 2E(r)}\mathrm{d}r^2 - R^2(t,r)\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2\right), \tag{15.16}$$

where $R(t, r)$ is determined by (15.14). It was first found and interpreted by Lemaître in 1933 [91]. Then, more of its properties were discussed in illuminating ways by Tolman in 1934 [141] and by Bondi in 1947 [142]. It will be called here the **Lemaître–Tolman (L–T) model**.[49] The L–T models were further discussed in many papers. Ref. [99] contains an overview of these, complete until 1994, but new papers keep being published.

The mass c^2M/G in (15.14) and (15.15) is the *active gravitational mass* that generates the gravitational field. It is in general different from c^2N/G – the sum of masses of particles that formed the gravitating body, compare eqs. (11.114) – (11.116). Suppose that matter fills the interior V of a sphere with the centre at the centre of symmetry of the space, $r = r_c$, and the surface at $r = r_S$. Then c^2N/G is, from (15.16):

$$c^2N/G = \int_V \left(\epsilon/c^2\right)\sqrt{-g}\ \mathrm{d}_3V \equiv 4\pi \int_{r_c}^{r_S} \frac{\left(\epsilon/c^2\right) R^2 R,_r}{\sqrt{1 + 2E(r)}}\ \mathrm{d}r, \tag{15.17}$$

while the active gravitational mass M is, from (15.15)

$$c^2M/G = 4\pi \int_{r_c}^{r_S} \left(\epsilon/c^2\right) R^2 R,_r\, \mathrm{d}r. \tag{15.18}$$

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[141] R. C. Tolman, *Proc. Nat. Acad. Sci. USA* **20**, 169 (1934); reprinted in *Gen. Relativ. Gravit.* **29**, 935 (1997), with an editorial note and author's biography by A. Krasiński, *Gen. Relativ. Gravit.* **29**, 931 and 932 (1997).

[142] H. Bondi, *Mon. Not. Roy. Astr. Soc.* **107**, 410 (1947); reprinted in *Gen. Relativ. Gravit.* **31**, 1783 (1999), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **31**, 1777 (1999) and author's (auto)biography by H. Bondi, *Gen. Relativ. Gravit.* **31**, 1780 (1999).

[49] This solution is called here "Lemaître–Tolman" only to avoid confusion with the Friedmann–Lemaître models, otherwise there would be no reason to add anybody's name to Lemaître's. Nevertheless, the name that is most frequently used in literature for this model is "Lemaître–Tolman–Bondi" (LTB).

When $E > 0$ M is larger than N, when $E < 0$ M smaller than N, when $E = 0$, $M = N$. If $M < N$, then $(N - M)$ is called the **relativistic mass defect** – the analogue of the mass defect known from nuclear physics. In a bound system, where $E < 0$, part of the energy contained in the component particles was shed, and this lost energy is responsible for the mass defect. In the opposite case ($E > 0$), the system is unbound and its excess energy sums up with the energy equivalent to the sum of masses of components. When $E = 0$, the system is "marginally bound" – no energy has been shed to form it, and there is no excess energy. This interpretation of E was one of the results of the Bondi paper [142].

With $\Lambda \neq 0$, the explicit solutions of (15.14) involve elliptic functions. They were discussed by Lemaître [91] and Omer [143]. When $\Lambda = 0$, they are as follows:

For $E(r) < 0$:

$$\begin{aligned} R(t,r) &= -\frac{M}{2E}(1 - \cos\eta), \\ \eta - \sin\eta &= \frac{(-2E)^{3/2}}{M}\left[t - t_B(r)\right]; \end{aligned} \tag{15.19}$$

For $E(r) = 0$:

$$R(t,r) = \left\{\frac{9}{2}M(r)\left[t - t_B(r)\right]^2\right\}^{1/3}; \tag{15.20}$$

For $E(r) > 0$:

$$\begin{aligned} R(t,r) &= \frac{M}{2E}(\cosh\eta - 1), \\ \sinh\eta - \eta &= \frac{(2E)^{3/2}}{M}\left[t - t_B(r)\right]. \end{aligned} \tag{15.21}$$

Actually, the conditions for each case are $E/M^{2/3} <, =$ or > 0 because regularity at the centre of symmetry requires that $E = 0$ at $M = 0$ in all cases; see Sec. 15.4.

The Friedmann solutions (14.33) – (14.35) follow from (15.19) – (15.21) as the limit

$$M = M_0 r^3, \qquad E = -\frac{1}{2}kr^2, \qquad t_B = t_0 = \text{constant}. \tag{15.22}$$

It follows then from (15.19) – (15.21) that $R(t,r) = rS(t)$, and (15.16) reduces to (14.3).

When M is constant, (15.15) shows that the L–T model becomes a vacuum solution of the Einstein equations. If $\Lambda = 0$, then this is the Schwarzschild solution in the Lemaître – Novikov coordinates, (11.106) – (11.107). In the following, we assume $M,_r \neq 0$.

Eq. (15.22) is coordinate-dependent. We now show that the invariant condition for the Friedmann limit is $\epsilon,_r = 0$. In each interval of r in which $M,_r$ does not change sign, $M(r)$ can be used as the independent variable, since (15.15) and (15.16) are covariant under the transformations $r = f(r')$. Using M as the radial coordinate, (15.15) becomes

$$\kappa\epsilon = \frac{6}{\left(R^3\right),_M} \Longleftrightarrow R^3 - R^3\left(M_0\right) = \int_{M_0}^{M} \frac{6}{\kappa\epsilon(\widetilde{M})}\, \mathrm{d}\widetilde{M}. \tag{15.23}$$

[143] G. C. Omer, *Proc. Nat. Acad. Sci. USA* **53**, 1 (1965).

Then, $\epsilon,_r = 0$ implies $\epsilon,_M = 0$ and $(R^3),_{MM} = 0$. This is equivalent to

$$E/M^{2/3} = \text{constant}, \qquad t_B = \text{constant}, \tag{15.24}$$

and these two equations define the Friedmann limit invariantly (see Exercise 3). The functions $\left(E/M^{2/3}\right),_M$ and $t_{B,M}$ generate two kinds of perturbation of the Friedmann models: the increasing and the decreasing one, respectively [144, 11].

The function $E(r)$ has one more interpretation. In a subspace $t =$ constant of (15.16), R depends only on r, so it can be used as the independent variable. With $(x^1, x^2, x^3) = (r, \vartheta, \varphi)$, we find the components of the 3-dimensional Riemann tensor to be:

$$R_{1212} = \frac{R_{1313}}{\sin^2\vartheta} = -\frac{RE,_R}{1+2E}, \qquad R_{2323} = -2ER^2\sin^2\vartheta. \tag{15.25}$$

Thus, with $E = 0$, every space $t =$ constant is flat, and $E \neq 0$ is a measure of the curvature of these subspaces. Unlike in the Friedmann models, that curvature is *local* – it depends on r, and in particular may be positive in one neighbourhood of the space, but negative elsewhere. This shows that the prominent role of the curvature index k is a peculiarity of the R–W class, and not a property of the physical Universe – the same spacetime can be approximately like the $k > 0$ Friedmann model in one neighbourhood and like the $k < 0$ Friedmann model in another. Thus, (15.19) – (15.21) do not really describe different cosmological models – they can hold in different regions of the same spacetime.

15.4 Conditions of regularity at the centre.

Not the whole set $R = 0$ is a singularity. Part of it is the centre of symmetry, and we will now formulate the conditions that the arbitrary functions have to obey in order that this centre remains nonsingular.[50]

Let $r = r_c$ at the centre of symmetry, where $R(t, r_c) = 0$ for all $t > t_B(r_c)$, and let $\epsilon(t, r_c)/c^2 = \rho(t, r_c) \overset{\text{def}}{=} \alpha(t) < \infty$. Assuming that $R,_r$ is also finite at $r = r_c$, (15.18) implies

$$M(r_c) = 0 \tag{15.26}$$

(no mass-point at the centre). Applying the de l'Hôpital rule we then find using (15.23)

$$\lim_{r\to r_c}\frac{R}{M^{1/3}} = \left[\lim_{r\to r_c}\frac{R^3}{M}\right]^{1/3} = \left[\lim_{r\to r_c}3R^2R,_M\right]^{1/3} = \left[\frac{6}{\kappa\epsilon(t, r_c)}\right]^{1/3}. \tag{15.27}$$

[144] J. Silk, *Astron. Astrophys.* **59**, 53 (1977).

[50] One could consider models with a permanently existing singularity at the centre of symmetry, but so far they have not been proven relevant for astrophysics. But there exist models in which $R \neq 0$ everywhere apart from the BB singularities, which means they have no centres – they are the Datt – Ruban models [45, 46] and their generalisations:
[145] V. A. Ruban, in: *Tezisy dokladov 3-y Sovetskoy Gravitatsyonnoy Konferentsii* [*Theses of Lectures of the 3rd Soviet Conference on Gravitation*]. Izdatel'stvo Erevanskogo Universiteta (1972), Erevan, p. 348;
[146] V. A. Ruban, *ZhETF* **85**, 801 (1983); English translation: *Sov. Phys. JETP* **58**, 463 (1983);
[147] P. Szekeres, *Commun. Math. Phys.* **41**, 55 (1975);
see also [137]. The spaces $t =$ constant in these models are 3-dimensional analogues of cylinders.

Thus, with $0 < \epsilon(t, r_c) < \infty$, R must behave in the neighbourhood of $r = r_c$ as

$$R = \beta(t) M^{1/3} + O_{1/3}(M), \qquad \beta = \left(\frac{6}{\kappa \alpha c^2}\right)^{1/3}. \tag{15.28}$$

where the symbols $O_a(M)$ will denote quantities with the property

$$\lim_{M \to 0} \frac{O_a(M)}{M^a} = 0. \tag{15.29}$$

Equation (15.28) is always fulfilled for the $E = 0$ model. For the other two models, defining

$$\gamma \stackrel{\text{def}}{=} \frac{1}{2}\, {\beta,_t}^2 - \frac{1}{\beta} + \frac{1}{6}\, \Lambda \beta^2 \tag{15.30}$$

we obtain from (15.15) $\gamma =$ constant and

$$E = \gamma M^{2/3} + O_{2/3}(M). \tag{15.31}$$

Finally, from (15.20) and (15.21), we see that $t_B(r_c)$ must have a finite value, i.e.,

$$t_B = \tau + O_0(M). \tag{15.32}$$

15.5 Formation of voids in the Universe.

Voids are large (approx. 60 Mpc in radius) volumes in the intergalactic space with a very low matter density. Their observational discovery at the end of the 1970s [148] was a surprise because it contradicted the then-universal belief that galaxies are distributed uniformly in space. In fact, the first papers indicating that voids should be ubiquitous were published in 1934 by Tolman [141] and Sen [149], but had not been understood.

Tolman's main result was the proof that the Einstein and Friedmann models are unstable against the growth of inhomogeneities. This is proved as follows. Let the initial conditions at $t = t_1$ be chosen so that $R_{LT}(t_1, r)$ in the L–T model is the same function as $rR_F(t_1)$ in the Friedmann model and $R_{LT,t}(t_1, r) = rR_{F,t}(t_1)$. Since the radial coordinate has not yet been fixed, these conditions do not reduce the L–T model to Friedmann. From the equations assumed it follows that $\left(R,_{tr}/R,_r\right)(t_1)\big|_{\rm LT} = \left(R,_t/R\right)(t_1)\big|_{\rm F}$. Since $R,_t$ is a measure of the velocity of expansion, the above implies that Tolman considered a perturbation of initial density in the Friedmann model, with unperturbed initial velocity.

From (15.15) we find

$$\left[\frac{\partial^2}{\partial t^2}\, \ln \epsilon\right]_{LT}(t_1) = \left[-2\frac{R,_{tt}}{R} + 2\frac{{R,_t}^2}{R^2} - \frac{R,_{ttr}}{R,_r} + \frac{{R,_{tr}}^2}{{R,_r}^2}\right]_{LT}(t_1)\,. \tag{15.33}$$

[148] S. A. Gregory and L. A. Thompson, The Coma/A1367 supercluster and its environs. *Astrophys. J.* **222**, 784 (1978).

[149] N. R. Sen, *Z. Astrophysik* **9**, 215 (1934); reprinted in *Gen. Relativ. Gravit.* **29**, no 11, 1477 (1997), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **29**, 1473 (1997) and author's biography by A. K. Raychaudhuri, *Gen. Relativ. Gravit.* **29**, 1474 (1997).

Using (15.14) to find $R,_{tt}$ and $R,_{ttr}$, then using (15.15), we simplify (15.33) to

$$\left[\frac{\partial^2}{\partial t^2}\ \ln\epsilon\right]_{LT}(t_1) = \left[\frac{1}{2}\kappa\epsilon + \Lambda + 2\frac{R,_t{}^2}{R^2} + \frac{R,_{tr}{}^2}{R,_r{}^2}\right]_{LT}(t_1)\,. \tag{15.34}$$

A similar equation holds for the Friedmann models:

$$\left[\frac{\partial^2}{\partial t^2}\ \ln\epsilon\right]_{F}(t_1) = \left[\frac{1}{2}\kappa\epsilon + \Lambda + 3\frac{R,_t{}^2}{R^2}\right]_{F}(t_1)\,. \tag{15.35}$$

Subtracting (15.35) from (15.34) and using the assumptions we find:

$$\frac{\partial^2}{\partial t^2}\ (\ln\epsilon_{LT} - \ln\epsilon_F) = \frac{1}{2}\kappa\left(\epsilon_{LT} - \epsilon_F\right). \tag{15.36}$$

This shows that wherever $\epsilon_{LT} - \epsilon_F \neq 0$, the difference in densities will be increasing in time. This means that an L–T model with initial condensations *or voids* will be evolving away from the background Friedmann model. In Tolman's own words:

"... at those values of r where the density in the distorted model is different from that in the Friedmann model, there is at least an initial tendency for the differences to be emphasised ... in cases where condensation is taking place ... the discrepancies will continue until we reach a singular state involving infinite density or reach a breakdown in the simplified equations."

Sen carried out a complementary study – he assumed the initial density to be unperturbed, with the initial velocity distribution being non-Friedmannian. By a similar method as Tolman, he concluded that "the models are unstable for initial rarefaction".

Ref. [99] contains a complete overview of studies of void formation done on the basis of the L–T model until 1994. The most complete study is contained in the papers by Sato and coworkers [150, 151, 152, 153], extensively summarised by Sato [154].

15.6 Formation of other structures in the Universe.

Bonnor [155] was the first to investigate formation of galaxies using an L–T model. He considered a Friedmann tube around the centre of symmetry, surrounded by an L–T transition zone, and that in turn surrounded by another Friedmann region. If both Friedmann regions have positive spatial curvature and the density in the inner one is higher than that in the outer one, then the inner region will start to recollapse earlier than the background and will form a condensation. Bonnor assumed that the condensation has the

[150] K. Maeda, M. Sasaki and H. Sato, *Progr. Theor. Phys.* **69**, 89 (1983).

[151] H. Sato and K. Maeda, *Progr. Theor. Phys.* **70**, 119 (1983).

[152] K. Maeda and H. Sato, *Progr. Theor. Phys.* **70**, 772 (1983).

[153] K. Maeda and H. Sato, *Progr. Theor. Phys.* **70**, 1276 (1983).

[154] H. Sato, in: *General Relativity and Gravitation.* Edited by B. Bertotti, F. de Felice and A. Pascolini. D. Reidel, Dordrecht, p. 289 (1984).

[155] W. B. Bonnor, *Z. Astrophysik* **39**, 143 (1956); reprinted in *Gen. Relativ. Gravit.* **30**, 1113 (1998), with an editorial note by A. Krasiński, *Gen. Relativ. Gravit.* **30**, 1111 (1998) and author's (auto)biography by W. B. Bonnor, *Gen. Relativ. Gravit.* **30**, 1112 (1998).

mass of a typical galaxy, that is, it contains $N \cong 3 \times 10^{67}$ nucleons, and that it formed at $t_i \cong 1000$ years after the BB. Then the following problem arose: if such a condensation formed as a statistical fluctuation in a homogeneous background, then initially $\delta\epsilon/\epsilon =\mid \epsilon_c - \epsilon_b \mid /\epsilon_b \cong N^{-1/2} \cong 10^{-34}$, where ϵ_c is the density in the condensation and ϵ_b is the background density. But in order to develop into a typical galaxy, the initial perturbation at t_i would have to be of the order $\delta\epsilon/\epsilon \cong 10^{-5}$. On the other hand, if a perturbation of the order of 10^{-5} is to arise as a statistical fluctuation, then it can involve only 10^{10} particles.

If the curvature is negative in the outermost region and positive in the innermost one, then the initial perturbation has to be ≈ 10 times larger than in the preceding case. If both Friedmann regions have negative curvature, a galaxy cannot form at all. The cosmological constant does not help if a model begins from a BB. Hence, two possibilities are left: either $\Lambda \neq 0$ and the Universe begins as an instability in the Einstein static Universe in the asymptotic past (then there is an arbitrary amount of time available for the statistical fluctuations to grow) or there exists a mechanism for producing large perturbations.

Bonnor's approach was exceptionally bold at that time, when most people avoided applying exact solutions of Einstein's equations to astrophysical problems, fearing great computational difficulties.

The current thinking is that the initial fluctuations of density were generated by quantum fluctuations of the scalar field that drives the inflation and indeed are of the order of 10^{-5} [156]. However, density fluctuations alone do not determine the structures: the velocity distribution at the initial time must be taken into account, too [112, 157, 158]. In particular, an initial condensation can evolve into a void [159]. In fact, any given (spherically symmetric) initial distribution of density or velocity can evolve into any other such distribution by means of an L–T model. With the initial distributions obeying the constraints imposed by the observations of the cosmic microwave radiation, the final distributions can be made to be realistic models of galaxy clusters, voids and galaxies with central black holes [112, 157, 160, 158].

15.7 The influence of cosmic expansion on planetary orbits

The first formally correct study of the problem of expansion of planetary orbits was carried out by Einstein and Straus [161].[51] They showed that a Schwarzschild region can be surrounded by a Friedmann spacetime, with the matching conditions fulfilled. The proof

[156] T. Padmanabhan, *Cosmology and Astrophysics Through Problems*, Cambridge U P, 1996.
[157] A. Krasiński and C. Hellaby, *Phys. Rev.* **D69**, 023502 (2004).
[158] C. Hellaby and A. Krasiński, *Phys. Rev.* **D73**, 023518 (2006).
[159] N. Mustapha and C. Hellaby, *Gen. Relativ. Gravit.* **33**, 455 (2001).
[160] A. Krasiński and C. Hellaby, *Phys. Rev.* **D69**, 043502 (2004).
[161] A. Einstein and E. G. Straus, *Rev. Mod. Phys.* **17**, 120 (1945); + erratum *Rev. Mod. Phys.* **18**, 148 (1946).

[51] The results of papers published earlier by other authors were coordinate-dependent [99].

given below is adapted to the notation used earlier in this course (Einstein and Straus' notation was rather outlandish).

Let us first consider matching the Schwarzschild metric, represented in the Lemaître – Novikov coordinates (11.106) – (11.107), to a general L–T model (15.16). Let the boundary hypersurface be $r = r_b$, with the Schwarzschild metric at $r \leq r_b$. We conclude from the matching conditions of Section 10.13 that the $R(t, r_b)$ in (11.106) must be the same (*as a function of* t) as the $R(t, r_b)$ in (15.16). Consequently, $R,_t(t, r_b)$ must be the same in both metrics, and then (15.14) implies that $E(r_b)$ is the same on both sides, and $m = M(r_b)$, where $M(r_b)$ is the L–T mass contained within the $r = r_b$ hypersurface of (15.16).

Now we take the Friedmann limit of (15.14) and (15.16). Hence, by (15.22) and (14.28), $M(r_b) = (G\mathcal{M}/c^2)\, {r_b}^3$, and the Friedmann limit of (15.16) is (14.3). Consequently

$$m = \frac{G\mathcal{M}{r_b}^3}{c^2} \stackrel{\text{def}}{=} \mu\,(r_b)\,. \tag{15.37}$$

This says that the Schwarzschild mass at the centre of symmetry is equal to the Friedmann mass removed from within the sphere $r = r_b$. The geodesic radius of that sphere measured from the Friedmann spacetime, $R(t)\int_0^{r_b}(1-kr^2)^{-1/2}\mathrm{d}r$, expands together with the Universe.

The fact that the Schwarzschild and Friedmann metrics can be matched implies that the planetary orbits are *in this configuration* not influenced by the expansion of the Universe. This result was for many years taken as the general implication of relativity. However, (15.37) need not be fulfilled if the Einstein–Straus configuration is taken only at a single moment $t = t_0$ as an initial condition for an L–T model. The results of other papers ([154] and papers cited therein, [162]) imply that if $m < \mu(r_b)$, then the boundary of the vacuole will expand faster than the Friedmann background, whereas if $m > \mu(r_b)$, then initial conditions may be set up so that the vacuole will start to collapse. This indicates that the Einstein–Straus configuration is unstable against perturbations of the condition (15.37), that is, it is an exceptional situation.

The same problem was studied by a different method by Gautreau [163, 11]. He based his study on the subcase $E = 0$ of the L–T model that he derived in a hybrid of comoving and curvature coordinates, in which the metric is non-diagonal. In these coordinates, R is the radial coordinate defined by the curvature radius of the orbits of the symmetry group. These orbits do not participate in the cosmic expansion and therefore R of any single orbit can be used as a standard of length. He assumed that the L–T model extends throughout each space of constant t (no vacuole), but contains a concentration of mass around the centre of symmetry – a model of a star. We report here only the conclusions.

By investigating the equations of timelike geodesics in his metric, Gautreau showed that circular orbits do not exist. This is in fact a Newtonian phenomenon: in Gautreau's model the smoothed-out cosmic matter density extends throughout the planetary system, and, as a result of cosmic expansion, matter streams out of every sphere $R =$ const. Hence, each planet moves under the influence of a gravitational force that is decreasing with time, so the orbit must spiral out. Gautreau derived the Newtonian formula for the rate of change of orbital radius, $\mathrm{d}R/\mathrm{d}t = 8\pi R^4 H\overline{\rho}/(2\mu)$, where R is the orbital radius, H is the

[162] K. Lake and R. Pim, *Astrophys. J.* **298**, 439 (1985).
[163] R. Gautreau, *Phys. Rev.* **D29**, 198 (1984).

Hubble parameter and $\overline{\rho}$ is the mean cosmic density of matter. The effect is thus greater for orbits of greater radius; for Saturn it is $(\mathrm{d}R/\mathrm{d}t)_S = 6 \times 10^{-18}$ m per year. This is obviously unmeasurable (one proton diameter per 1000 years!). For a star at the edge of the Andromeda galaxy the effect would be $(\mathrm{d}R/\mathrm{d}t)_{\mathrm{gal}} = 1100$ km per year. Gautreau's result thus implies that the reaction of planetary orbits to the expansion of the Universe is not measurable with any technology available today. However, it is important to know that in principle the effect is nonzero – if the hydrodynamical description of matter in the Universe is consistently applied down to small scales.

15.8 Apparent horizons in the L–T model.

As defined in Sec. 13.4, the apparent horizon (AH) is the outer envelope of the region of closed trapped surfaces, while a closed trapped surface S_t is one from which it is impossible to send a diverging bundle of light rays – because both the outward-directed and inward-directed bundles immediately converge: ${k^\mu}_{;\mu} \leq 0$ at S_t.

We will find the apparent horizon in the L–T model for the centre of symmetry by the method of Szekeres [164]. For this purpose, it suffices to consider families of null geodesics sent orthogonally from a surface $r =$ constant. We must identify the surface at which ${k^\mu}_{;\mu}$ becomes zero for all future-directed null geodesics. Since the surface we are looking for is a sphere, its normal rays will be radial. From (15.16), the tangent vectors to radial null curves obey $k^0 - \varepsilon \left(R_{,r} / \sqrt{1 + 2E}\right) k^1 = 0$, $k^2 = k^3 = 0$, where $\varepsilon = +1$ for outward-directed curves and $\varepsilon = -1$ for inward-directed curves. Because of spherical symmetry, these curves must be geodesics. Now consider a bundle of null geodesics originating at a surface $S_{t,r}$ given by $\{t = t_s, r = r_s\}$. The affine parameter on these geodesics may be chosen so that

$$k^0 = \frac{R_{,r}}{\sqrt{1 + 2E}}, \qquad k^1 = \varepsilon \qquad \text{on } S_{t,r}. \tag{15.38}$$

The divergence of this field on $S_{t,r}$ is then

$$\begin{aligned} {k^\mu}_{;\mu} &= {k^\mu}_{,\mu} + \begin{Bmatrix} \mu \\ \rho\nu \end{Bmatrix} k^\rho k^\nu \\ &= {k^0}_{,t} + {k^1}_{,r} + \frac{R_{,r}}{\sqrt{1 + 2E}} \left(\frac{R_{,tr}}{R_{,r}} + 2\frac{R_{,t}}{R}\right) + \varepsilon \left(-\frac{E_{,r}}{1 + 2E} + \frac{R_{,rr}}{R_{,r}} + 2\frac{R_{,r}}{R}\right). \end{aligned} \tag{15.39}$$

Since k^μ is null, geodesic and affinely parametrised, we have $k_\mu k^\mu = 0$ and ${k^\mu}_{;\nu} k^\nu = 0$. We differentiate the first equation by t and take the second equation with the index $\mu = 1$, then we take both results on $S_{t,r}$:

$${k^0}_{,t} - \frac{R_{,tr}}{\sqrt{1 + 2E}} - \varepsilon \frac{R_{,r}}{\sqrt{1 + 2E}} {k^1}_{,t} = 0, \tag{15.40}$$

$$\frac{R_{,r}}{\sqrt{1 + 2E}} {k^1}_{,t} + \varepsilon {k^1}_{,r} + 2\varepsilon \frac{R_{,tr}}{\sqrt{1 + 2E}} + \frac{R_{,rr}}{R_{,r}} - \frac{E_{,r}}{1 + 2E} = 0. \tag{15.41}$$

[164] P. Szekeres, *Phys. Rev.* **D12**, 2941 (1975).

We add (15.41) multiplied by ε to (15.40), and thereby eliminate $k^1{}_{,t}$. The resulting equation is used to eliminate $k^0{}_{,t} + k^1{}_{,r}$ from (15.39), and then $k^\mu{}_{;\mu} = 0$ becomes

$$2\frac{R_{,r}}{R}\left(\frac{R_{,t}}{\sqrt{1+2E}} + \varepsilon\right) = 0. \tag{15.42}$$

One solution of this equation is $R_{,r} = 0$, but this is either a shell crossing singularity or a neck – for what happens there see Ref. [11]. The generic solution of (15.42) is

$$\frac{R_{,t}}{\sqrt{1+2E}} = -\varepsilon. \tag{15.43}$$

Thus, $R_{,t}$ has the sign of $-\varepsilon$. For outward-directed geodesics $\varepsilon = +1$, so the solution of (15.43) exists only in collapsing models. For inward-directed geodesics $\varepsilon = -1$, and the solution of (15.43) exists only in expanding models. Using (15.14), we have in (15.43)

$$\sqrt{2E + \frac{2M}{R} - \frac{1}{3}\Lambda R^2} = \sqrt{1+2E}. \tag{15.44}$$

With $\Lambda = 0$, the solution of this is

$$R = 2M. \tag{15.45}$$

In the Schwarzschild limit, M = constant, the apparent horizon becomes identical to the event horizon. In a general L–T spacetime, the r = constant shell obeying (15.45) is just falling into its own Schwarzschild horizon.

Inside the future apparent horizon, all light rays proceed toward the final singularity. The existence of such a region was predicted by Bondi [142], and later by Barnes [165]. In every collapsing L–T model, the dust matter must enter the future apparent horizon before it hits the final singularity at $R = 0$, and in every expanding L–T model the dust matter remains inside the past apparent horizon for a while after leaving the BB.

We now quote a few results from Refs. [160, 11] without proof, just for reference.

Hyperbolic regions (those with $E \geq 0$) either permanently expand or permanently collapse, i.e. only one AH can occur (either the future AH$^+$ or the past AH$^-$). Two AHs can thus cross only in an elliptic region (where $E < 0$). At the moment of maximum expansion, $R_{,t} = 0$, the maximal value of R obeys $2M/R_{\max} + 2E = 0$. Since $-1 < 2E < 0$ in elliptic regions, $R_{\max} > 2M$, so each mass shell gets out of AH$^-$ before it falls into AH$^+$. However, at those locations where $2E = -1$, the maximal R equals $2M$, which means that AH$^-$ touches AH$^+$. Such a location is a *neck*, the nonvacuum analogue of the Kruskal – Szekeres $r < 2m$ region.

The centre of symmetry, $M = 0$, is the only place where the AH can touch the BB or Big Crunch (BC). But whether this coincidence is real or only a coordinate effect depends on the shape of the functions $E(r)$ and $t_B(r)$. A null or timelike segment of the BB or BC singularity can stick out at the centre, and it is invisible in the comoving coordinates because they, too, have a singularity there that squeezes this segment into a point. A signature of this situation is a family of different light cones with vertices at the same comoving (r, t) [157, 166].

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[165] A. Barnes, *J. Phys.* **A3**, 653 (1970).

[166] P. S. Joshi, *Global Aspects in Gravitation and Cosmology.* Clarendon Press, Oxford 1993.

15.9 The redshift.

To use the general formula for redshift, (13.23), the field of tangent vectors to the light ray, k^α, must be affinely parametrised. Transforming it to such a parametrisation may be not easy, so, for numerical calculations, it is often more convenient to use other methods.

The following method was introduced by Bondi [142]). From (15.16), for a radial null geodesic proceeding toward the observer, we get

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{R,_r(t,r)}{\sqrt{1+2E(r)}}. \tag{15.46}$$

Let two light rays be emitted in the same direction, the second one later by a short time-interval τ. Let the equation of the first ray be $t = T(r)$, and that of the second ray $t = T(r) + \tau(r)$. Both rays must obey (15.46), so

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{R,_r(T(r),r)}{\sqrt{1+2E(r)}}, \qquad \frac{\mathrm{d}(T+\tau)}{\mathrm{d}r} = -\frac{R,_r(T(r)+\tau(r),r)}{\sqrt{1+2E(r)}}. \tag{15.47}$$

Since $\tau(r)$ was assumed small, we have, to first order in τ

$$R,_r(T(r)+\tau(r),r) = R,_r(T(r),r) + \tau(r)R,_{tr}(T(r),r). \tag{15.48}$$

Using (15.48) and the first of (15.47) in the second of (15.47) we get

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\tau(r)\frac{R,_{tr}(T(r),r)}{\sqrt{1+2E(r)}}. \tag{15.49}$$

If τ is the period of the light wave, then $\tau(r_{\rm obs})/\tau(r_{\rm em}) = 1 + z(r_{\rm em})$. Keeping the observer at a fixed position and considering the sources at two distances $r_{\rm em}$ and $r_{\rm em} + \mathrm{d}r$, we find that $(\mathrm{d}\tau/\mathrm{d}r)/\tau = -(\mathrm{d}z/\mathrm{d}r)/(1+z)$. Using this in (15.49):

$$\frac{1}{1+z}\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{R,_{tr}(T(r),r)}{\sqrt{1+2E(r)}}. \tag{15.50}$$

Hence, the redshift may be calculated numerically from:

$$\ln(1+z(r)) = \int_{r_{\rm obs}}^{r_{\rm em}} \frac{R,_{tr}(T(r),r)}{\sqrt{1+2E(r)}}\mathrm{d}r. \tag{15.51}$$

This formula is equivalent to (13.23) (see Exercise 4), but here the parameter on the null geodesic is just the coordinate r.

15.10 The blueshift

The possible existence of blueshifts in L–T models was casually mentioned without proof by P. Szekeres in a conference report in 1980 [167]. His note seems to imply that infinite

[167] P. Szekeres, in: *Gravitational Radiation, Collapsed Objects Exact Solutions.* Edited by C. Edwards. Springer (Lecture notes in physics, vol. 124), New York, p. 477 (1980).

blueshift will appear along every light ray that is emitted from a nonconstant segment of the BB, where $\mathrm{d}t_B/\mathrm{d}r \neq 0$. But there is another necessary condition, that the ray be directed radially (Szekeres considered only radial rays!) [168], which seems to have been overlooked by all later authors. We will verify it now.

The general equations defining the tangent vectors $k^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\lambda$ to geodesics of the metric (15.16), with λ being the affine parameter, are

$$\frac{\mathrm{d}k^t}{\mathrm{d}\lambda} + \frac{R_{,r}\, R_{,tr}}{1+2E}\,(k^r)^2 + RR_{,t}\,\left[\left(k^\vartheta\right)^2 + \sin^2\vartheta\,\left(k^\varphi\right)^2\right] = 0, \tag{15.52}$$

$$\frac{\mathrm{d}k^r}{\mathrm{d}\lambda} + 2\frac{R_{,tr}}{R_{,r}}\,k^t k^r + \left(\frac{R_{,rr}}{R_{,r}} - \frac{E_{,r}}{1+2E}\right)\,(k^r)^2 - \frac{(1+2E)R}{R_{,r}}\,\left[\left(k^\vartheta\right)^2 + \sin^2\vartheta\,\left(k^\varphi\right)^2\right] = 0, \tag{15.53}$$

$$\frac{\mathrm{d}k^\vartheta}{\mathrm{d}\lambda} + 2\frac{R_{,t}}{R}\,k^t k^\vartheta + 2\frac{R_{,r}}{R}\,k^r k^\vartheta - \cos\vartheta\sin\vartheta\,(k^\varphi)^2 = 0, \tag{15.54}$$

$$\frac{\mathrm{d}k^\varphi}{\mathrm{d}\lambda} + 2\frac{R_{,t}}{R}\,k^t k^\varphi + 2\frac{R_{,r}}{R}\,k^r k^\varphi + 2\frac{\cos\vartheta}{\sin\vartheta}\,k^\vartheta k^\varphi = 0. \tag{15.55}$$

The geodesics determined by (15.52) – (15.55) are null when

$$\left(k^t\right)^2 - \frac{{R_{,r}}^2\,(k^r)^2}{1+2E} - R^2\left[\left(k^\vartheta\right)^2 + \sin^2\vartheta\,(k^\varphi)^2\right] = 0. \tag{15.56}$$

Using $R_{,t}\,k^t + R_{,r}\,k^r = \mathrm{d}R/\mathrm{d}\lambda$ and $k^\vartheta = \mathrm{d}\vartheta/\mathrm{d}\lambda$, the general solution of (15.55) is

$$R^2\sin^2\vartheta k^\varphi = J_0, \tag{15.57}$$

where J_0 is constant along the geodesic.

Using the above, the general solution of (15.54) is

$$R^4\left(k^\vartheta\right)^2\sin^2\vartheta + {J_0}^2 = C^2\sin^2\vartheta, \tag{15.58}$$

where C^2 is another constant along the geodesic. When $C = 0$, the geodesic is radial. Then $J_0 = 0$ and either (a) $\vartheta = 0$ or π, with φ being undetermined or (b) ϑ is constant and φ is constant in consequence of (15.57).

From (15.57) and (15.58) we get $\left(k^\vartheta\right)^2 + \sin^2\vartheta\,(k^\varphi)^2 = C^2/R^4$, and then (15.56) becomes

$$\left(k^t\right)^2 = \frac{{R_{,r}}^2\,(k^r)^2}{1+2E} + \frac{C^2}{R^2}. \tag{15.59}$$

One can rescale the affine parameter λ to obtain on past-directed rays

$$k^t(t_o) = -1. \tag{15.60}$$

In comoving coordinates $u^\alpha = {\delta^\alpha}_0$, so, using (15.60), we obtain from (13.23)

$$1 + z = -k^t(t_e). \tag{15.61}$$

[168] C. Hellaby and K. Lake, *Astrophys. J.* **282**, 1 (1984); erratum *Astrophys. J.* **294**, 702 (1985).

For nonradial rays, on which $C \neq 0$, the last term in (15.59) will go to infinity when $R \to 0$. Thus, at the BB, independently of whether the penultimate term in (15.59) is finite or not

$$\lim_{R\to 0} \left|k^t\right| \equiv \lim_{R\to 0} z = \infty. \tag{15.62}$$

Consequently, a necessary condition for infinite blueshifts from the BB ($\lim_{R\to 0} z = -1$) is that the ray be radial. Whether this is also a sufficient condition has not so far been proved analytically, there exist only numerical calculations confirming it [169, 170].

15.11 General properties of the Big Bang/Big Crunch singularities in the L–T model

At all points apart from the centre of symmetry, both these singularities are spacelike. This is verified as follows: since the singularities are parts of the $R = 0$ set, calculate the normal vector field n^μ to any hypersurface $R =$ constant. It has components $(R_{,t}\,, R_{,r}\,, 0, 0)$, so $g^{\mu\nu} n_\mu n_\nu = g^{00} {R_{,t}}^2 + g^{11} {R_{,r}}^2$, which, using (15.14) and (15.16), becomes

$$g^{\mu\nu} n_\mu n_\nu = \frac{2M}{R} - 1. \tag{15.63}$$

This is positive for $R < 2M$. Consequently, n^μ is timelike there, i.e. the hypersurface $R =$ constant is spacelike. But as we approach the nonsingular centre of symmetry $R \to 0$, the regularity condition (15.28) implies that $2M/R \to 0$, so the nonsingular part of the set $R = 0$ is timelike. As we approach the singularity $R \to 0$ along a line with $M > 0$, $2M/R \to +\infty$, so the part of $R = 0$ in which $M > 0$ is spacelike.

The orientation of the central part of the singularity, where the line of centre of symmetry hits the BB or BC (so both $R \to 0$ and $M \to 0$), is not easy to determine, since the comoving coordinates have a singularity there, too. It had taken quite some time to notice that this set is not always a single point, but may be a finite segment of a timelike or null curve. The first to note this possibility (during a numerical investigation of an $E = 0$ model) were Eardley and Smarr [171], and they called this phenomenon **shell focussing**. They stated that the segment of the centre of symmetry in which shell focussing occurs can only be null. However, their criteria for the occurrence of a shell focussing, which they gave without derivation, do not look entirely credible because they refer to the limit of t_B/M as $R \to 0$, while the value of t_B is coordinate-dependent. C. Hellaby, in his unpublished PhD Thesis,[52] found that the segment in question can be timelike as well.

If the BB or BC singularity is all spacelike, then its intersection with the worldline of the centre of symmetry can be imagined as a single point – no future-directed light ray can leave it. However, if the singularity contains a timelike or null segment, then that segment

[169] A. Krasiński, *Phys. Rev.* **D90**, 103525 (2014).
[170] A. Krasiński, *Phys. Rev.* **D93**, 043525 (2016).
[171] D. M. Eardley and L. Smarr, *Phys. Rev.* **D19**, 2239 (1979).

[52] See abbreviated accounts of the results of this Thesis in Ref. [11] and in the unpublished preprint
[172] C. Hellaby and K. Lake, *The singularity of Eardley, Smarr and Christodoulou.* Preprint 88/7, Institute of Theoretical Physics and Astrophysics, University of Cape Town 1988.

contains vertices of an infinite family of distinct light cones. The conclusion is that the non-spacelike segment of the singularity is an extended arc of a curve that is mapped into a single point in the comoving coordinates [11, 172].

15.12 Increasing and decreasing density perturbations.

Formation of structures in the Universe used to be described by linearised perturbations of the Robertson–Walker models. In that approach, two classes of perturbations had been identified: those that increase and those that decrease with time. The same distinction exists in the L–T and Szekeres models [173]). For the L–T model with $\Lambda = 0$, this approach was first applied by Silk [144] and then presented, by a somewhat different method, in Ref. [11]. Since the details of the calculations are complicated, we only report the results here.

In an $E > 0$ model, when $t_{B,r} = 0$, the perturbation of the Friedmann density vanishes at the initial singularity and tends to a finite limit as $t \to \infty$, i.e. it is increasing. The perturbation of curvature, $[(2E)^{3/2}/M]_{,r}$, displays the exactly opposite behaviour. The perturbations generated by $t_{B,r} \neq 0$ are decreasing.

In an $E < 0$ model with $t_{B,r} = 0$, the gradient of $[(-2E)^{3/2}/M]$ generates a perturbation of the Friedmann model that vanishes at the initial singularity and increases without limits at the final singularity. With $E < 0$ and $[(-2E)^{3/2}/M]_{,r} = 0$, the density contrast tends to infinity at both singularities.

15.13 Mimicking the accelerating expansion of the Universe by an inhomogeneous mass distribution

The accelerating expansion of the Universe was deduced from observations of the type Ia supernovae. It was *assumed* that for all of them the absolute luminosity at peak is the same. The observed luminosities were inconsistent with the predictions of the $\Lambda = 0$ Friedmann model. For other Friedmann models, the best consistency with observations was achieved when $k = 0$, 32% of the energy density comes from matter (visible or dark) and 68% of the energy density comes from "dark energy", which plays the role of Λ [85]. This shows that the conclusion about accelerated expansion of the Universe followed from the *interpretation of observations via a pre-assumed Friedmann model*, so it is not an objectively registered fact. The example given below [58, 59, 174, 175] shows how the accelerating expansion can be imitated using an L–T model, without introducing "dark energy".

By a reasoning closely analogous to that which led to (14.71) we find that in an L–T geometry, the observer area distance r_o and the luminosity distance D_L between the observer at $R = 0$ and a light source at (t_e, r_e) are $R(t_e, r_e)|_{\rm ray}$ and $D_L = (1+z)^2\ R(t_e, r_e)|_{\rm ray}$,

[173] S. W. Goode and J. Wainwright, *Phys. Rev.* **D26**, 3315 (1982).

[174] C.-M. Yoo, T. Kai, K-i. Nakao, Redshift drift in Lemaître – Tolman – Bondi void universes. *Phys. Rev.* **D83**, 043527 (2011).

[175] A. Krasiński, *Phys. Rev.* **D89**, 023520 (2014); **D89**, 089901(E) (2014).

respectively. To calculate D_L as a function of z we need to know the functions $t(r)$ and $z(r)$ along the ray. The first can be found (numerically) as a solution of the null geodesic equation in the metric (15.16) with (15.14). Once this is done, the function $z(r)$ can be found from the Bondi formula (15.51). We then equate the $D_L(z)$ for the L–T model found in this way to the $D_L(z)$ for the ΛCDM model, eq. (14.78):

$$(1+z)^2 R(t(z), r(z)) = \frac{1+z}{\mathcal{H}_0} \int_0^z \frac{\mathrm{d}z'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}, \tag{15.64}$$

where $\Omega_m = 0.32$, $\Omega_\Lambda = 0.68$ and $c\mathcal{H}_0 = H_0 = 67.1$ km/(s × Mpc) are taken from the current observations [85]. Let us assume $E/r^2 = C = \text{constant} > 0$, as in Friedmann models (the solution of (15.64) exists for every C). Then (15.64) implicitly defines $t_B(r)$ via the evolution equations (15.21), so $t_B(r)$ can be numerically calculated.

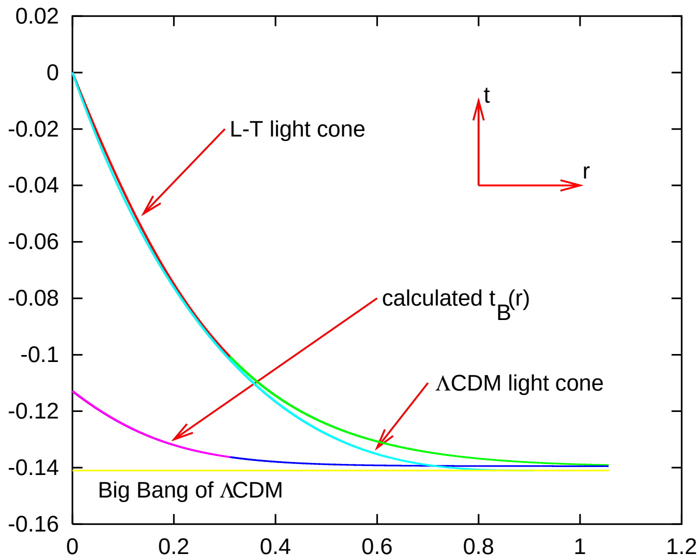

Figure 15.1: The past light cone of the present central observer in the L–T model that reproduces the $D_L(z)$ from (14.78) *using only* $t_B(r)$. The past light cone of the ΛCDM model with the same $D_L(z)$ is shown for comparison.

The $t_B(r)$ thus calculated, together with the assumed $E(r)$, determine the L–T model that has the same function $D_L(z)$ for the central observer as the ΛCDM model, eq. (14.78). Fig. 15.1 shows the comparison of the past light cones of the two models for the same present central observer; it also shows the corresponding BB graphs: the constant one for ΛCDM, and the $t_B(r)$ calculated from (15.64). The $t_B(r)$ graph should asymptotically approach the constant t_B of ΛCDM. The gap between them in the figures is a consequence of the numerical code becoming unstable in a close neighbourhood of the BB.

Figure 15.2 shows how the L–T model from Fig. 15.1 accounts for the apparent accelerated expansion. In the L–T model, the BB occurs progressively later when the observer position is approached. At the point P the particle in the L–T model is "younger" than in a Friedmann model whose constant t_B asymptotically approaches $t_B(r)$ (call it the "right"

t_B for brevity). The age difference increases toward the observer. Therefore, the particle in the L–T model is less decelerated than it would be in any $\Lambda = 0$ Friedmann model that has the right t_B. Consequently, its expansion velocity at P is larger than it would be in that Friedmann model, and the difference in velocities increases toward the observer. This means that, instead of increasing with time, the expansion velocity increases, relative to the Friedmann model, on approaching the position of the observer. The conclusion is: if an L–T model were used to interpret the observations, no "accelerating expansion" would appear and there would be no need to introduce the "dark energy".

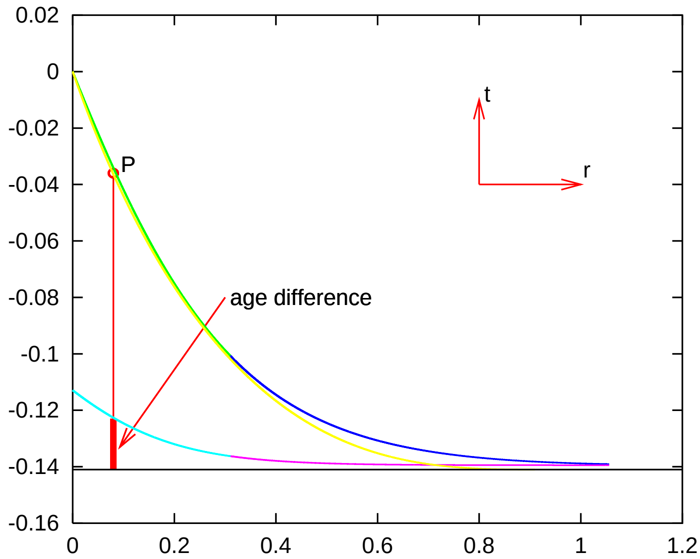

Figure 15.2: The light cones and the $t_B(r)$ function from Fig. 15.1. The vertical straight line is the world line of a particle of the cosmic fluid (these world lines are vertical straight lines because r is a comoving coordinate). At P the world line intersects the past light cone of the present central observer, which nearly coincides with the ΛCDM past light cone there. The thick vertical segment shows the difference in age for the particle observed at P between the ΛCDM and L–T models.

The model discussed above did away with the dark energy by using just the function $t_B(r)$; the function $E(r)$ was assumed the same as in a Friedmann model. But the dark energy can be explained away also by assuming t_B constant and adapting $E(r)$ to fulfil (15.64) [176]. By allowing both $E(r)$ and $t_B(r)$ to be freely adaptable, one can explain away the acceleration and fit the L–T model to one more set of data [177].

The models presented above are too simple to be fitted to the whole set of observations of the type Ia supernovae. They are meant to be a warning that the idea of dark energy may have no basis in reality, similarly to the (now long-forgotten) ideas of ether and continuous creation of matter in steady-state models of the Universe.

[176] A. Krasiński, *Phys. Rev.* **D90**, 023524 (2014).
[177] M.-N. Célérier, K. Bolejko and A. Krasiński, *Astron. Astrophys.* **518**, A21 (2010).

15.14 Drift of light rays

This topic will be presented here briefly and only qualitatively; see Ref. [178] for details.

Null geodesics cannot have constant r over any open segment because (15.53) shows that then also ϑ and φ would be constant, so such a geodesic would be timelike. At isolated points though, $\mathrm{d}r/\mathrm{d}\lambda = 0$ is possible. Thus r can be used as a (non-affine) parameter along every segment of a null geodesic on which $\mathrm{d}r/\mathrm{d}\lambda \neq 0$. Imagine two *nonradial* light rays in an L–T spacetime, the second one emitted later by τ by the same source, both arriving at the same observer. Let the trajectory of the first ray be

$$(t, \vartheta, \varphi) = (T(r), \Theta(r), \varPhi(r)). \tag{15.65}$$

Unlike for radial geodesics, we cannot assume that the second ray will proceed through the same values of Θ and $\varPhi$ as the first one. So, the equation of the second ray will be

$$(t, \vartheta, \varphi) = (T(r) + \tau(r), \Theta(r) + \zeta(r), \varPhi(r) + \psi(r)). \tag{15.66}$$

Already at this point, without any calculations, we may conclude the following: The second ray intersects each given hypersurface $r = r_0$ not only later, but in general at a different comoving location. Consequently, the two rays intersect different sequences of matter worldlines between the source and the observer. The second ray will thus be received by the observer from a different direction in the sky.

It follows that an observer in an L–T spacetime should see a generic light source drift across the sky. The drift vanishes only for radial directions. The spacetimes in the L–T family in which there is no drift for all observers and all rays are the Friedmann models [178]. Observational detection of the drift would thus be evidence of inhomogeneity of the Universe on large scales.[53]

Figures 15.3 and 15.4 show a numerical example of the drift. The parameters of this configuration are only illustrative, they do not reflect the properties of any real object. The rays propagate through a void of radius ≈ 7 Gpc, the mass density at the centre of the void is $\approx 1/5$ of the average background density, the observer and the light source are at ≈ 3.5 Gpc from the centre of the void, the density profile is $\rho(t_0, x) = \rho_0 \left[1 + \delta - \delta \exp\left(-x^2/\sigma^2\right)\right]$, where t_0 is the current instant, x is defined as $R(t_0, r)$, ρ_0 is the density at the origin, $\delta = 4.05$ and $\sigma = 2.96$ Gpc. The graph of this density (profile 1) is shown in the right panel of Fig. 15.3. The projections of three rays on the space $t =$ now along the dust flow lines are shown in Fig. 15.4.

The drift phenomenon has not yet been worked out for any observed object. Preliminary estimates [178] imply that the rate of drift within a void should be of the order of 10^{-6} arc seconds per year. With the accuracy of the existing Gaia observatory, which is $5 - 20 \times 10^{-6}$ arcsec [181], a few years of monitoring a given source would be needed to detect this effect.

[178] A. Krasiński and K. Bolejko, *Phys. Rev.* **D83**, 083503 (2011).

[53] For a geometric description of the drift in a general spacetime see Refs. [179, 180].

[179] M. Korzyński and J. Kopiński, *J. Cosm. Astropart. Phys.* **03**, 012 (2018).
[180] M. Grasso, M. Korzyński and J. Serbenta, *Phys. Rev.* **D99**, 064038 (2019).
[181] http://sci.esa.int/science-e/www/area/index.cfm?fareaid=26

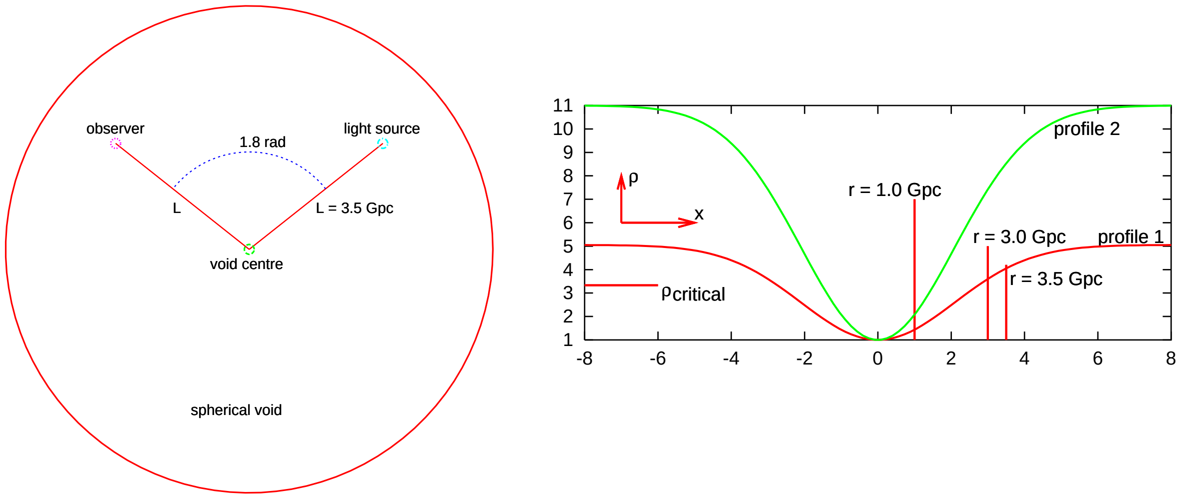

Figure 15.3: **Left panel:** The L–T configuration used in the example. The present distance from the centre of the void to the observer and to the light source is $L = 3.5$ Gpc. **Right panel:** The density profile used in the calculation for Fig. 15.4 is profile 1.

15.15 Exercises.

1. Prove that if (1) The spacetime is spherically symmetric, and (2) The metric obeys the Einstein equations with a perfect fluid source, then the rotation is necessarily zero.

2. Prove that $\dot{u}^\alpha = 0$ in (15.1) – (15.2) implies $C,_r = 0$.

3. Prove that eqs. (15.24) are the necessary and sufficient conditions for the L–T model to reduce to the Friedmann limit.

Hint: Use M as the radial coordinate. As observed in (15.23), these conditions are equivalent to $(R^3),_{MM} = 0$. Calculate $(R^3),_{MM}$ separately for each sign of E. With $E = 0$, the result $t_{B,M} = 0$ follows quite easily. In the other two cases observe that $\partial(M, \eta)/\partial(M, t) = \eta,_t \neq 0$. Hence, M and η can be chosen as the independent variables. Eliminate $(t - t_B)$ from $(R^3),_{MM}$ using (15.19) and (15.21). What will result is a polynomial in functions of η with coefficients depending on M only. Equate to zero the coefficients of independent functions of η; if you do it in the right order, (15.24) will follow quite simply. In order to show that (15.24) indeed define the Friedmann model, change the radial variable to $r = (M/M_0)^{1/3}$ and substitute whatever you have found in (15.16), then compare the result with (14.1) and (14.32) – (14.34).

4. Prove that the formula for redshift, eq. (15.51), is equivalent to (13.23).

Hint: With the coordinates of (15.16), eq. (13.23) can be written as

$$1 + z = (k^0)_e/(k^0)_o. \tag{15.67}$$

Keeping the observer at fixed position, take the logarithmic derivative of this by r and

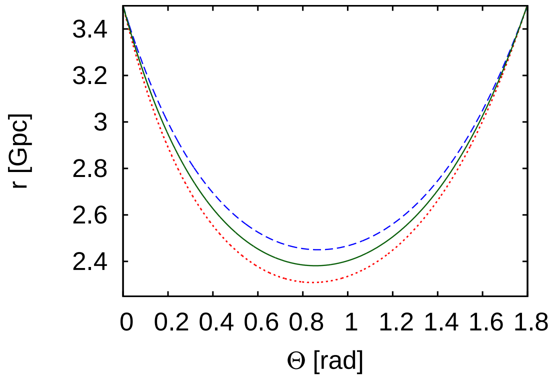

Figure 15.4: Rays proceeding from the source to the observer in Fig. 15.3 projected on the space $t =$ now along the dust flow lines. **Middle line:** the ray received now. **Upper line:** the ray received 5×10^9 years ago. **Lower line:** the ray received 5×10^9 years in the future.

compare the result with (15.50). The conclusion is

$$\frac{(k^0)_{,r}}{k^0} = \frac{R_{,tr}\left(T(r), r\right)}{\sqrt{1 + 2E(r)}}. \tag{15.68}$$

Integrating this and using the equation of an ingoing radial null geodesic, we get

$$k^0 = \mathrm{e}^{\int \frac{R_{,tr}(T(r),r)}{\sqrt{1+2E(r)}} \mathrm{d}r}, \qquad k^1 = -\frac{\sqrt{1+2E}}{R_{,r}}\, \mathrm{e}^{\int \frac{R_{,tr}(T(r),r)}{\sqrt{1+2E(r)}} \mathrm{d}r}, \qquad k^2 = k^3 = 0. \tag{15.69}$$

Now it can be verified that this field obeys $k^\alpha{}_{;\beta}\, k^\beta = 0$, and so is affinely parametrised. In verifying this, note that the expression under the integral does not depend on t – it is an integral calculated along the null geodesic, where the integrand is a function of r only.

5. Let the mass density in the L–T model (15.15) be a differentiable function of r. Prove that the spatial extrema of density in the $E \neq 0$ models are comoving only when the following equations are fulfilled at the extremal values:

$$\begin{aligned}
&\left(\frac{|2E|^{3/2}}{M}\right)_{,r} = \left(\frac{|2E|^{3/2}}{M}\right)_{,rr} = t_{B,r} = t_{B,rr} = 0,\\
&\left[\left(\frac{M^3}{E^3}\right)_{,r} \frac{1}{M_{,r}}\right]_{,r} = 0.
\end{aligned} \tag{15.70}$$

Find the corresponding condition for the $E = 0$ model.

Note: The fact that the extrema are comoving only under the conditions (15.70) means that in general they are not comoving. Thus, they move across the flow lines of dust. This phenomenon is known in cosmology as **density waves**. Their existence in relativistic models was predicted by Ellis, Hellaby and Matravers [182] by methods of the linearized Einstein theory. Note also that the first four conditions mean that at the extremum the L–T model has a second-order contact with the Friedmann model that approximates it.

[182] G. F. R. Ellis, C. Hellaby and D. R. Matravers, *Astrophys. J.* **364**, 400 (1990).

Hint. Choose M as the spatial coordinate. The first four conditions do not change then, but the last one takes the more readable form $(M^3/E^3)_{,MM} = 0$. The extrema are at those values of M where $\epsilon_{,M} = 0$, i.e. $(R^3)_{,MM} = 0$. Then proceed as in Exercise 3. If the extrema are comoving, then the values of M obeying (15.70) do not depend on t.

Chapter 16

Relativistic cosmology IV: The Szekeres models

This class of generalisations of the Friedmann models has interesting geometry and offers a wealth of astrophysical applications, but it involves rather complicated calculations. Therefore, we shall only introduce the metric and skip its derivation and a study of its properties.

P. Szekeres[54] took the following Ansatz for the metric [147]:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2\alpha}\mathrm{d}r^2 - \mathrm{e}^{2\beta}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{16.1}$$

where α and β are functions of (t, r, x, y) to be determined from the Einstein equations. The source is dust, $p = 0$, and the coordinates of (16.1) are comoving so that $u^\mu = {\delta^\mu}_0$. Under these assumptions, Szekeres found all solutions of Einstein's equations. In general, they have no symmetry (all Killing vector fields are zero).

There are a few sub-families of these metrics. We consider here only the quasi-spherical Szekeres (QSS) solutions, which generalise the L–T models. They have the metric [183]

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2 - \frac{\mathcal{E}^2 (\Phi/\mathcal{E}),_r{}^2}{1+2E(r)}\mathrm{d}r^2 - \frac{\Phi^2}{\mathcal{E}^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \\ \mathcal{E} &\overset{\text{def}}{=} \frac{(x-P)^2}{2S} + \frac{(y-Q)^2}{2S} + \frac{S}{2}, \end{aligned} \tag{16.2}$$

where $E(r)$, $M(r)$, $P(r)$, $Q(r)$ and $S(r)$ are arbitrary functions, and $\Phi(t, r)$ obeys

$$\Phi,_t{}^2 = 2E(r) + \frac{2M(r)}{\Phi} - \frac{1}{3}\Lambda\Phi^2. \tag{16.3}$$

This is the same evolution equation that governed the Friedmann and the L–T models, (14.25) and (15.14), and again it implies that the BB time is in general position-dependent:

[54] Peter Szekeres, the discoverer of this class of models, is a son of György Szekeres, the co-discoverer of the Kruskal–Szekeres coordinates for the Schwarzschild metric discussed in Sec. 11.7.

[183] C. Hellaby, *Class. Quant. Grav.* **13**, 2537 (1996).

$$\int_0^{\Phi} \frac{\mathrm{d}\widetilde{\Phi}}{\sqrt{2E + 2M/\widetilde{\Phi} + \frac{1}{3}\Lambda\widetilde{\Phi}^2}} = t - t_B(r). \tag{16.4}$$

The mass density is

$$\kappa\rho = \frac{2\,(M/\mathcal{E}^3)\,{}_{,r}}{(\Phi/\mathcal{E})^2\,(\Phi/\mathcal{E})\,{}_{,r}}, \qquad \kappa = \frac{8\pi G}{c^2}. \tag{16.5}$$

The surfaces of constant t and r are nonconcentric spheres, see Fig. 16.1, x and y are stereographic coordinates on them. The mass-density distribution (16.5) is that of a mass dipole (!) superposed on a spherical mass monopole [184]. The functions $P(r)$, $Q(r)$ and $S(r)$ tell where the dipole axis pierces a given sphere of constant t and r. In the axially symmetric case shown in Fig. 16.1, $P = Q \equiv 0$. For illustrations showing a nonsymmetric family of nonconcentric spheres in a Szekeres spacetime see Ref. [185].

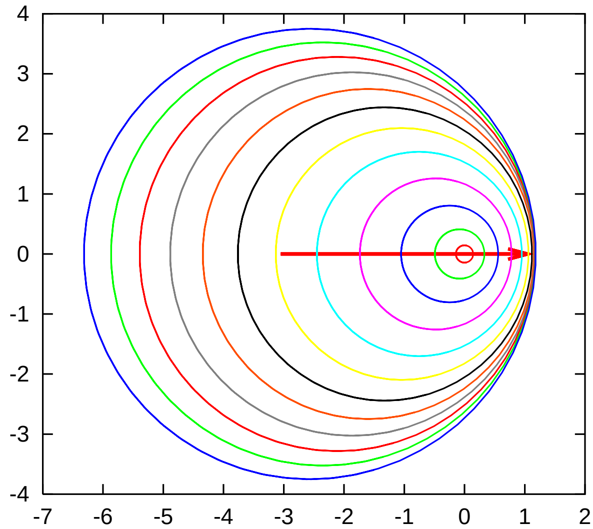

Figure 16.1: A cross-section through the constant-r spheres in a space of constant t in a quasi-spherical Szekeres metric. The family of spheres was drawn as axially symmetric, but in general it is not. The arrow points along the direction of the dipole maxima. See text for more explanations.

The L–T model is contained here as the limit of constant (P, Q, S) – then the spheres become concentric and the spacetime becomes spherically symmetric. The Friedmann limit follows when, in addition, $\Phi(t, r) = rR(t)$, $2E = -kr^2$ where $k =$ constant is the Friedmann curvature index and t_B is constant.

For more general information on the Szekeres models see Ref. [11], for a recent application to cosmology see Ref. [186],

[184] M. M. de Souza, *Rev. Bras. Fis.* **15**, 379 (1985).
[185] A. Krasiński, *Phys. Rev.* **D94**, 023515 (2016).
[186] R. A. Sussman, I. D. Gaspar, *Phys. Rev.* **D92**, 083533 (2015).