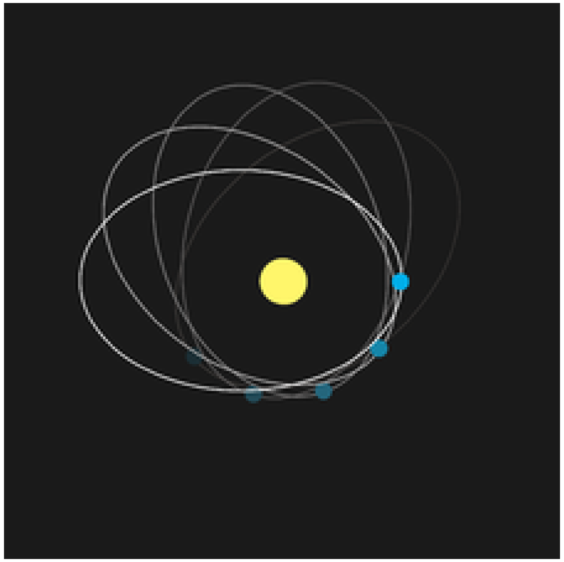
The motion of a test particle in gravitational field of a black hole

I. General Relativity course in 15 minutes

When Albert Einstein formulated his General Relativity theory w 1915 r. it was rather difficult to find application for that theory. Clear predictions included only the precession of the Mercury's orbit and the light bending close to the Sun.

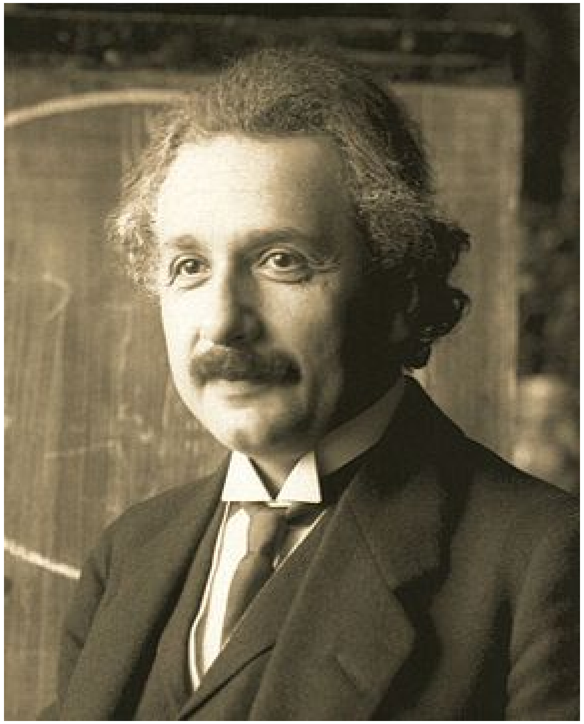


From wikipedia

Budget of precession in arc sec/century:

532.3035 – from other planets
0.0286 – from solar oblatness
42.9799 - GR effect

575.31 - total predicted
574.10±0.65 - total observed (in 1947)



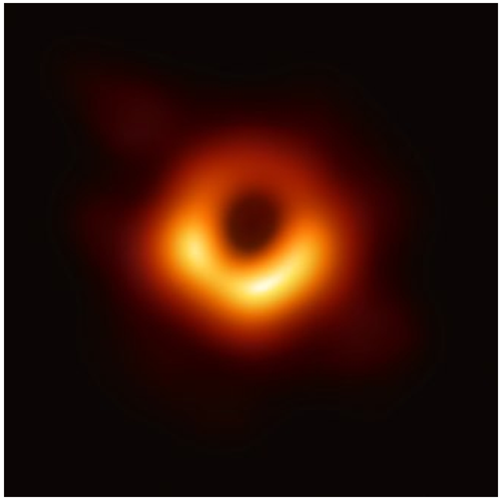
Albert Einstein in 1921 (wikipedia). According to ADS, his most cited paper is Einstein, Podolsky & Rosen, 1935, „Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? - **6987** *citations*
Best cited GR paper(from 1916) – 871 citations

The motion of a test particle in gravitational field of a black hole

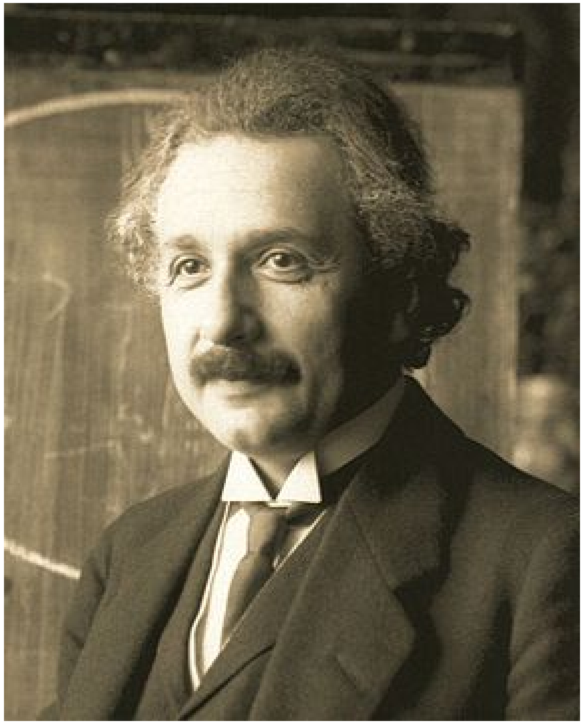
I. General Relativity course in 15 minutes

When Albert Einstein formulated his General Relativity theory w 1915 r. it was rather difficult to find application for that theory. Clear predictions included only the precession of the Mercury’s orbit and the light bending close to the Sun.

Now we cannot live comfortably without GR. We need GR in astronomy (cosmology, black holes in the cores of all galaxies) or even in everyday life (GPS).



M87 image from Event Horizon Telesope and Yacht GPS receiver (wikipedia)



Albert Einstein in 1921 (wikipedia). According to ADS, his most cited paper is Einstein, Podolsky & Rosen, 1935, „Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? - 6987 citations
Best cited GR paper(from 1916) – 871 citations

I. General Relativity course in 15 minutes

1. Time and space in classical mechanics

Time and space in classical mechanics do not mix up, any transformations of the reference frame preserve separately the time intervals
$dt = \text{const}$
and the space intervals
$dl^2 = dx^2 + dy^2 + dz^2 = \text{const}$
Where we used Cartesian coordinate system.

2. Time and space in special relativity

Time and space in special relativity are mixed up, creating a four-dimensional spacetime. If we change the coordinate system, only the spacetime intervals are preserved, not the time intervals and space intervals separately.
In Cartesian coordinates this spacetime distance reads

$ds^2 = c^2dt^2 – (dx^2 + dy^2 + dz^2) = \text{const}$
We will use quite frequently spherical coordinates, so in spherical coordinated the distance is

$$ds^2 = c^2\,dt^2 - (dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2)$$

The effects of the special relativity are important when the velocity (for example dx/dt) is close to the light speed, c. Otherwise the mixing is not important, space and time remain well separated.

2. Time and space in special relativity (cont.)

$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) = const$

Looking at this spacetime distance we see that ds^2 can be negative, and we have a clear separation in the distance character connected to photons.

Photons: $ds^2 = 0$
Timelike intervals: $ds^2 > 0$
Spacelike intervals: $ds^2 < 0$

Points having $ds^2 > 0$ can be connected with the worldline, i.e. a particle can travel from the starting point to reach the planned destination. The order of events along the line is fixed. In the case of spacelike intevals, the points cannot be connected with w worldline, and the order of events cannot be set, we do not know what happened earlier, and what happened later, the region is not causally connected. Coordinate transformation does not change ds, and thus the character of the distance.

3. General Relativity

This theory is based on the concept that gravity is equivalent to the spacetime curvature. Since the spacetime is curved, the distances are always more complicated to calculate

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$$

where the presence of the subscript and superscript implies the summation, and α and β run from 1 to 4.

3. General Relativity

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

Since the shape of the spacetime, according to Einstein, must be connected to matter, there are equations which connect the **metric tensor**, $g_{\alpha\beta}$, to the matter distribution

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Einstein's original equation

Law of an expanding universe — All matter and energy in the universe

$T_{\mu\nu}$ is the stress-energy tensor and describes the material sources of the gravity

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Law of an expanding universe — Cosmological constant — All matter and energy in the universe

$$G_{\mu\nu} = 8\pi G (T_{\mu\nu} - \bar{\rho}_{DE}\, g_{\mu\nu})$$

Law of an expanding universe — All matter and energy in the universe

Here $G_{\mu\nu}$ contains $g_{\mu\nu}$ and their first and second derivatives. I applologize for a change of notation from α,β to μ,ν...

3. General Relativity

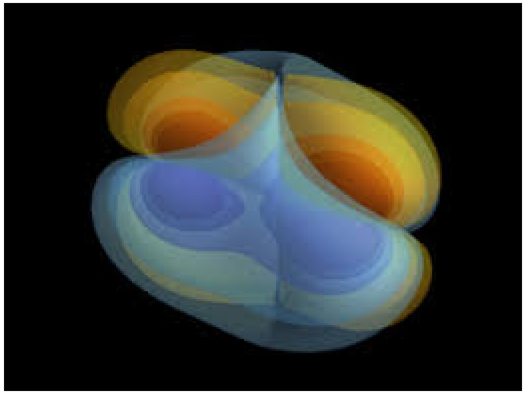
In this lacture we aim at describing the motion of a test particle around a black hole, so we forget about cosmology and cosmological constant, and we will limit outselves to classical form:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Law of an expanding universe

All matter and energy in the universe

Since $G_{\mu\nu}$ contains up to second derivatives of the metric tensor, it describes the curvature of the spacetime. If **T = 0 everywhere in the spacetime**, we have no curvature ad equations reduce just to Special Relaticivty case (flat Minkowski space). In general, equations are complicated (several non-linear equations), and the solution has to be found numerically. There are program to solve them, e.g. Einstein Toolkit.



Einstein Toolkit webpage, binary black hole merger

But there are a few simple cases, when the solution can be found analytically due to imposed assumptions. Such simple cases are:

(a) cosmology – Friedmann-Lemaitre-Robertson-Walker metric

This was derived first by Alexander Friedmann (Александр Александрович Фридман) in 1922, and analysed in detail in 1924. It was later rediscovered by others although the original publications of Friedmann were in German, so accesible to the world at that time...

(a) cosmology – Friedmann-Lemaitre-Robertson-Walker metric

The solution is based on assumption that the Universe is uniform and isotropic in space, and then the metric in the spherical coordinated has the form:

$$ds^2 \;=\; c^2\,dt^2 - R^2(t)\,[\frac{dr^2}{1-kr^2} + r^2\,d\,\theta^2 + r^2 \sin^2\theta\,d\,\phi^2]$$

If you compare it for the distance in flat space, but also in sphericall coordinates (page 3 of this lecture) where we had:

$$ds^2 \;=\; c^2\,dt^2 - (dr^2 + r^2\,d\,\theta^2 + r^2 \sin^2\theta\,d\,\phi^2)$$

You see that all the difference is in the scale factor R(t) in front of the spatial part (the Universe can be expanding or conctrating, so distance scale change), and the factor multiplying the radial distance element. If **$k=0$ the space it flat**, just expanding. If k is not equal zero, than a circle of a ring with small (!) length

$2\pi r$ has the radius: **$r(1+1/6kr^2)$**

if I calculated it correctly. If there is no cosmological constant, the space with negative curvature gives a closed model of the Universe, which will recollapse, while the space with negative curvature corresponds an open model, always expanding.

Currently the Universe seems flat, although with cosmological constant. All is coded in R(t) and the material filling the Universe.

(b) gravitational field around a point-like source

Next example whih is of key interest for today's lecture is the solution for point-like source of the mass M and zero angular momentum:

$$ds^2 = c^2\left(1 - \frac{R_{Schw}}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R_{Schw}}{r}} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

This solution was found in spherical coordinates since again in this case we have a spherical symetry. The gravitational field strenth here is coded through the parameter:

$$R_{Schw} = \frac{2GM}{c^2}$$

the Schwarzschild radius

This solution is asymptotically flat (no gravity at infinity), but close to the body M the space is strongly curved.

This solution represents either a point-like source (black hole), or a star of the radius R, when the radius of the star is larger than R_{Schw} corresponding to its mass.

In case of a star, the solution applies only outside the star, inside the star another solution has to be found which depends on the equation of state etc.

Karl Schwarzschild (wikipedia), He derived the solution in 1915, and died a year later

Digression: a dark star concept

A concept of 'black holes' is much older. John Mitchell in a paper for the Philosophical Transactions of the Royal Society of London, read on 27 November 1783, first proposed the idea that there were such things, which he called "dark stars".

That John Mitchell was an Irish nationalist activist, solicitor and political journalist !!!!

The concept is simple:

What if, at the star surface

Escape velocity = Light speed

In such case

$$E=\frac{1}{2}mv^2-\frac{GMm}{R^2}=0 \quad \Rightarrow \quad v^2=\frac{2GM}{R}$$

And for v = c we get the dark star radius

$$R_{dark\,star}=\frac{2GM}{c^2}$$

Just like for Schwarzschild radius.

But we get the motion of the photon incorrectly !

Since nothing escapes from more compact system, the Schwarzschild radius gives the black hole horizon (for a non-rotating black hole).

(b) gravitational field around a point-like source

Now, equiped with the proper GR solution, we can have an insight into what is happening close to the black hole horizon.

$$ds^2 = c^2\left(1 - \frac{R_{Schw}}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{R_{Schw}}{r}} - r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

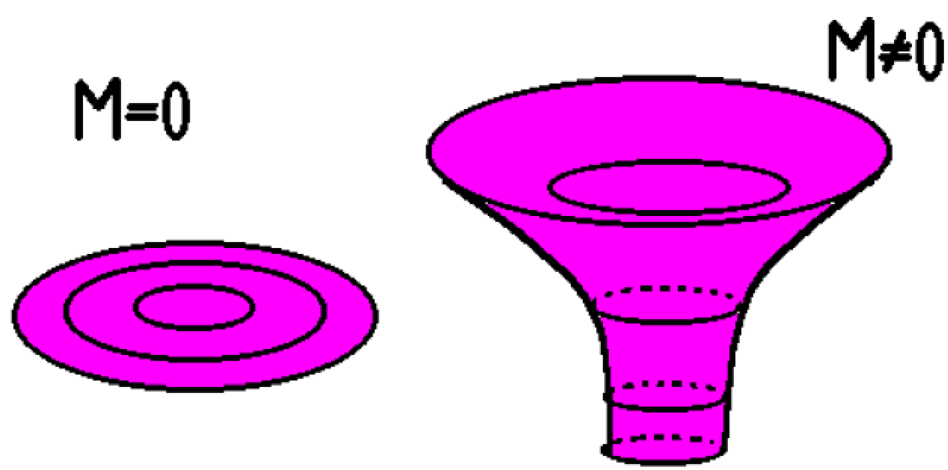
Locally (but only locally!) we can always introduce a coordinate system which corresponds to the Minkowski flat space which corresponds to a local observer:

$$d\tau = c\sqrt{1 - \frac{R_{Schw}}{r}}\, dt \qquad d\tilde{r} = \frac{dr}{\sqrt{1 - \frac{R_{Schw}}{r}}}$$

For this local observer time flows more slowly, and seems to stop at the horizon of a black hole. So if we observe from a distance a particle falling down onto the black hole, the particle seems to slow down and freeze, although in reality it speeds up and cross the horizon with a speed of light.

On the other hand, is seems like there is more space there; for a given value of r, the circle has the length 2πr, but the radial separation for this local observer is not dr, but dr_tilda, much larger than dr.

$$\frac{dr}{dt} = \left(1 - \frac{R_{Schw}}{r}\right)\frac{d\tilde{r}}{d\tau}$$



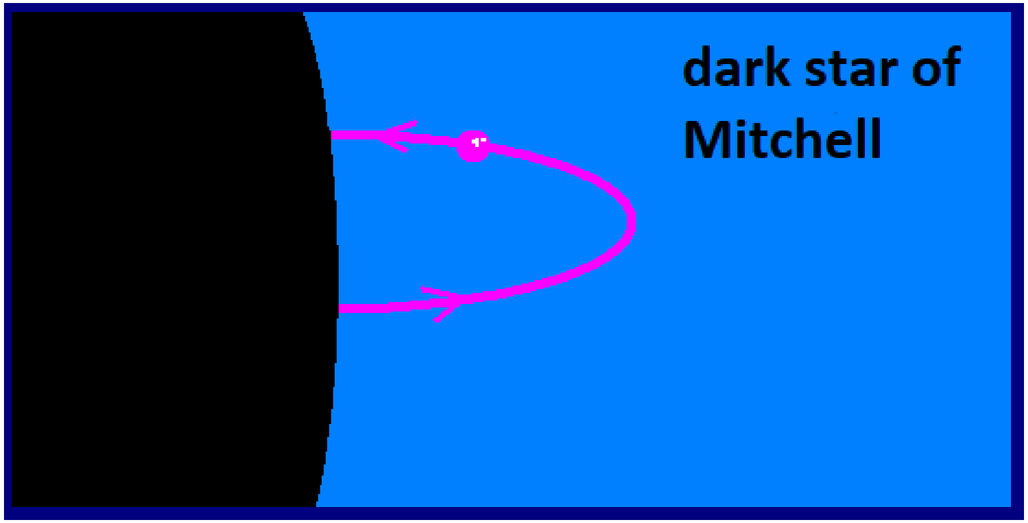
(b) gravitational field around a point-like source

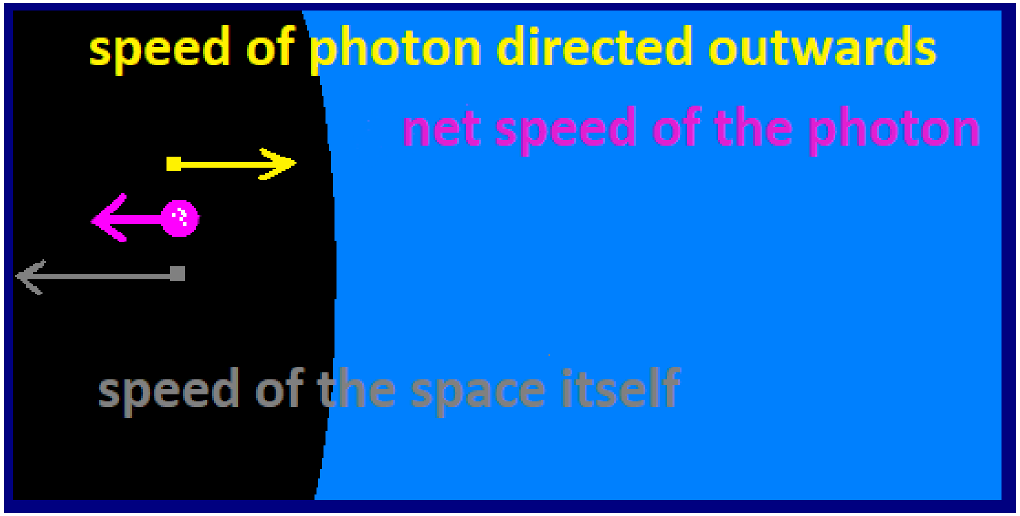
$$ds^2 = c^2\left(1-\frac{R_{Schw}}{r}\right)dt^2 - \frac{dr^2}{1-\frac{R_{Schw}}{r}} - r^2\left(d\theta^2+\sin^2\theta\, d\phi^2\right)$$

To see mathematically what is actually happening at the horizon, when the matter flows in, requires an introduction of another coordinate system which does not show the singularity, but this is possible, and there is no real singularity at the horizon r = R_{schw}. It took years to realize that the only real singularity is at r = 0. However, below the horizon the meaning of time and space swap

- Outside the particle moves up and down in radius, but in one direction in time
- Inside the partcle moves up and down in time but only in one direction in radius (inwards)

Intuitive explanation of a qualitative difference between the Newtonian concept of Mitchell and full GR description of black hole.





II. Equations of motion of a test particle in gravitational field

1. Classical mechanics: Newton theory

We will consider the motion of a test in gravitational field. Euler equations then simplify (no pressure etc.)

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \nabla \vec{v} = \cancel{-\nabla P} + \vec{f}$$

Here gravitational force is also proportional to the density, which disappears from equation. More importantly, gravity is a conservative force, depending only on the position (and not on velocity, as for example frictional force) so force can be represented as a gradient of the potential,

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \nabla \vec{v} = -\nabla \psi$$

Now we use the fact that the full time derivative and partial derivatives relate as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \nabla$$

we write the equation above in the compact form $\quad \frac{d \vec{v}}{dt} = -\nabla \psi$

Next we multiply this equation by $\vec{v}$ and we get $\quad \vec{v} \frac{d \vec{v}}{dt} = -\vec{v} \nabla \psi$

We can now formally calculate the full time derivative of the potential

$$\frac{d}{dt}\psi = \frac{\partial}{\partial t}\psi + \vec{v} \nabla \psi \quad \Longrightarrow \quad \frac{1}{2}\frac{d v^2}{dt} + \frac{d \psi}{dt} = 0$$

since the potential does not depend on time

1. Classical mechanics: Newton theory

$$\frac{1}{2}\frac{d\,v^2}{dt} + \frac{d\,\psi}{dt} = 0 \quad \Rightarrow \quad \frac{1}{2}v^2 + \psi = const = e$$

In this way we derived **the conservation of energy.**

We can derive in a similar way the **the conservation of angular momentum.**
We do that by multiplying the equation of motion not by velocity but by radius

$$\frac{d\,\vec{v}}{dt} = -\nabla\psi \quad / \times \vec{r} \longrightarrow \vec{r}\times\frac{d\,\vec{v}}{dt} = -\vec{r}\times\nabla\psi$$

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = \vec{v}\times\vec{v} + \vec{r}\times\frac{d\,\vec{v}}{dt}$$

On the right hand side first term is zero, second term is what we need for the equation, and if the potential is spherically symmetric (we will concentrate on such cases) we have

$$\nabla\psi \sim \vec{r}, \quad \Rightarrow \quad \vec{r}\times\nabla\psi = 0$$

And we get the requested conservation law:

$$\vec{r}\times\vec{v} = const = \vec{l}$$

Here the angular momentum is a vector, and the value is per unit mass.

1. Classical mechanics: Newton theory

The motion in a spherically symmetric case is constrained to a plane. Choosing the coordinated related to this plane, and introducing radial and asimuthal velocity components we can reduce the angular momentum law to a scalar condition

$$r\,v_{\phi} \;=\; l$$

And we can combine it with energy equation

$$\frac{1}{2}v_r^2 + \frac{1}{2}\frac{l^2}{r^2} + \psi \;=\; e$$

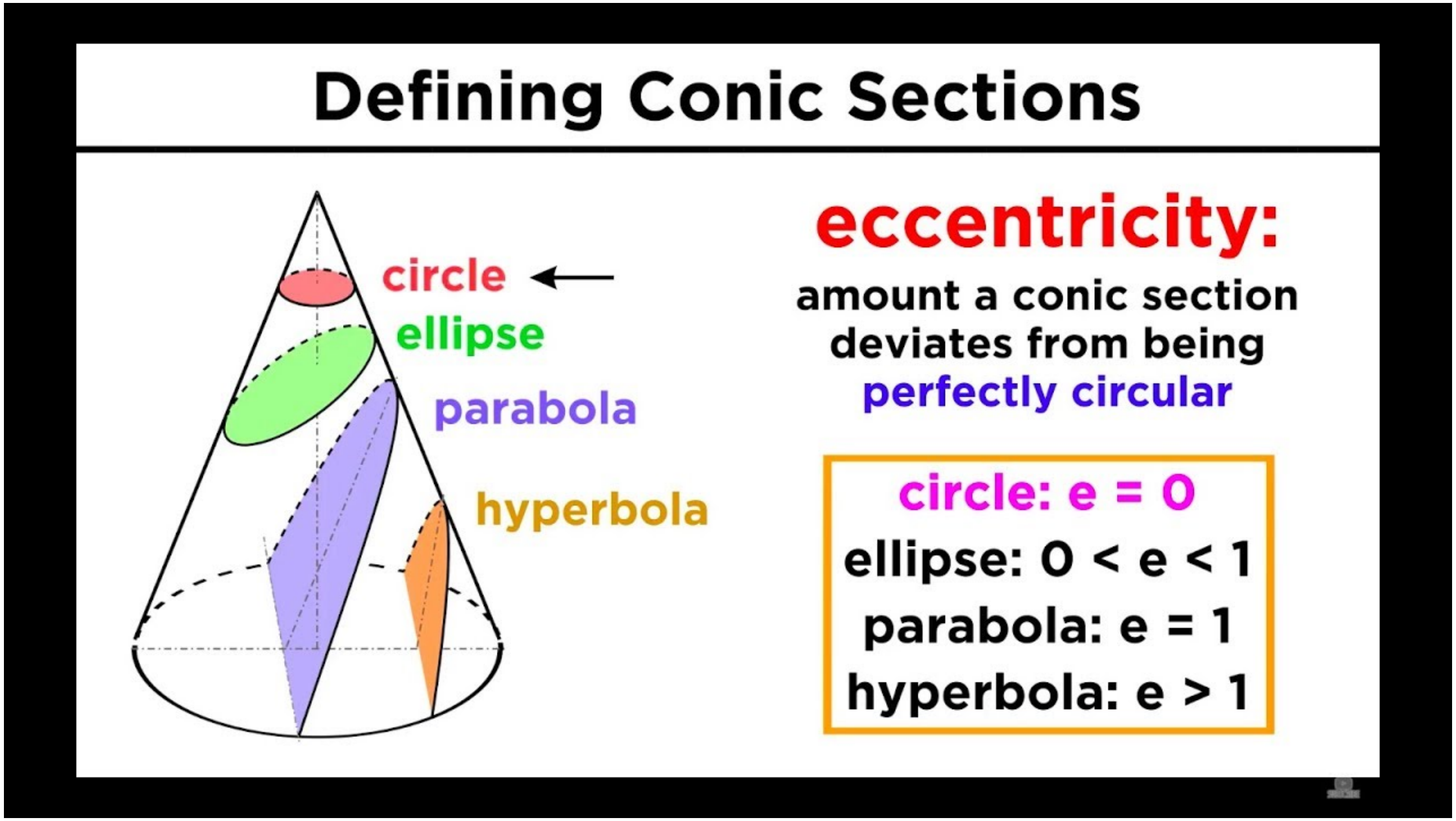


Illustration from YouTubehttps://www.youtube.com/watch?v=JUvo3GrgWHk

Now we will work just with these two equations. Orbits in general are of the conic section character, ellipse or hyperbola, with specific cases of a parabola and a circle.

We will concentrate on circles.

2. Circular orbits in the Newtonian case

$$\frac{1}{2}v_r^2+\frac{1}{2}\frac{l^2}{r^2}+\psi \;=\; e$$

To get a circular orbit, we have to request:

$v_r = 0$ and $dv_r/dr = 0$

The first condition is not enough since for an elliptical orbit vr = 0 appears in two places.
The second condition prevent the change of the value of the radial velocity.
In order to calculate circular orbits it is convenient to introduce the effective potential

$$\psi_{eff} \;=\; \psi+\frac{1}{2}\frac{l^2}{r^2}; \qquad \psi_{eff}^{Newton} \;=\; -\frac{GM}{r}+\frac{1}{2}\frac{l^2}{r^2}$$

So the conservation of energy now reads

$$\frac{1}{2}v_r^2+\psi_{eff}=e$$

and the two conditions for a circular orbit reduce to (i)

$$e=\frac{-GM}{r}+\frac{1}{2}\frac{l^2}{r^2}$$

(ii) $$\frac{d\psi_{eff}}{dr} \;=\; 0; \qquad \frac{d\psi_{eff}^{Newton}}{dr} \;=\; 0 \;=\; \frac{GM}{r^2}-\frac{l^2}{r^3} \rightarrow l_K \;=\; \sqrt{GMr},$$

And (i) combined with (ii) gives

$$e_K \;=\; -\frac{1}{2}\frac{GM}{r}$$

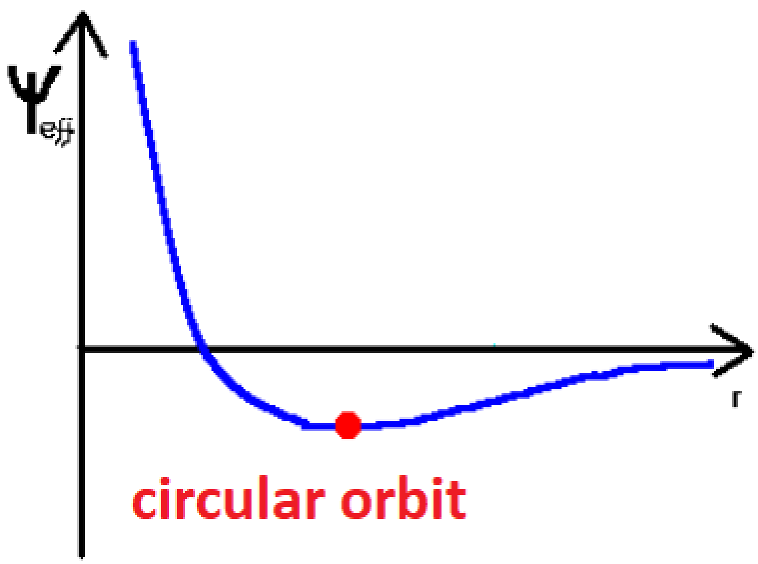
2. Circular orbits in the Newtonian case

Thus the circular (we frequently say Keplerian) orbit radius is set uniquely by the chosen value of the angular momentum:

$$l_K = \sqrt{GMr}$$

$$e_K = -\frac{1}{2}\frac{GM}{r}$$

And if the effective potential in plotted as a function of radius, for an assumed l_K, the minimum corresponds to the circular orbit. There is just one such orbit.



Now you probably wonder why we are doing such a stupid excercise. Everybody knows that.

3. Circular orbits in pseudo-Newtonian potential

Pseudo-Newtonian potential is a mathematical trick which allows to calculate the most important GR effects in Schwarzschild metric without actually learning full GR. It was introduced by Paczyński and Wiita in 1980.

The idea is simple:

$$\psi^{PNewton} = -\frac{GM}{r - R_{Schw}}$$

A photo of Bohdan Paczyński from a Biographical Memoir by Bruce T. Draine

The term $-R_{Schw}$ imitates the position of the horizon. So again we introduce the effective potential

$$\psi_{eff}^{PNewton} = \quad = -\frac{GM}{r - R_{Schw}} + \frac{1}{2}\frac{l^2}{r^2}$$

And again we differentiate the potential to get the position of the circular orbit. But now we get more complex result:

Paul Wiita, photo from https://science.tcnj.edu/2018/09/04/paul-wiita/

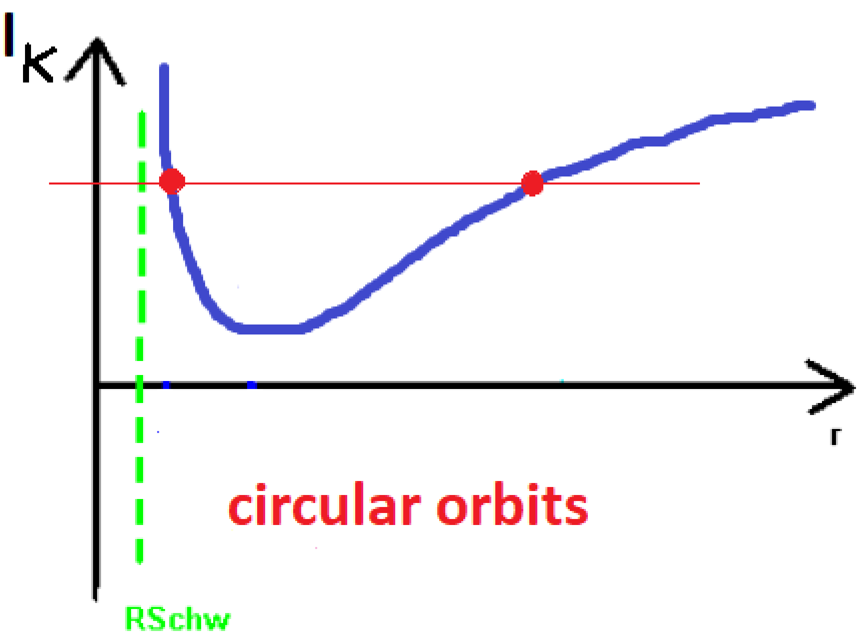
3. Circular orbits in pseudo-Newtonian potential

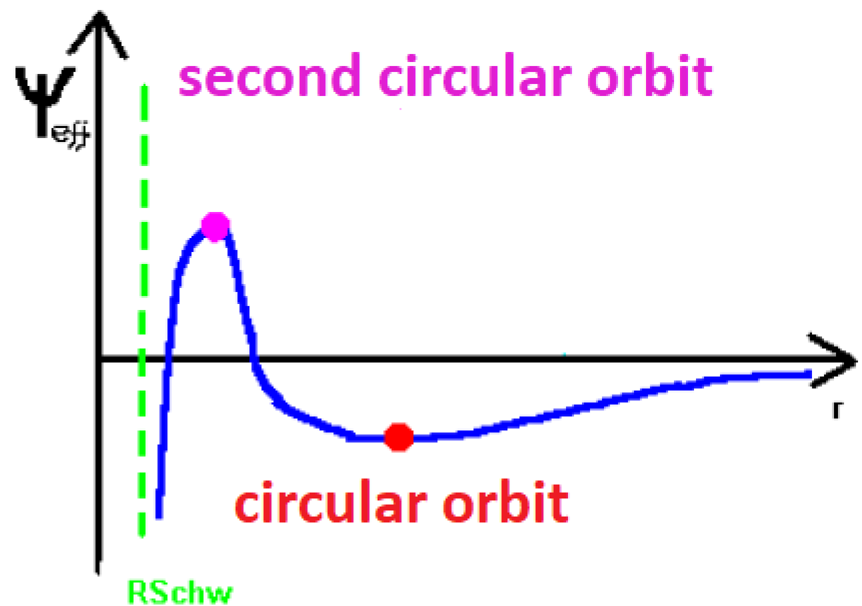
$$\psi^{PNewton} = -\frac{GM}{r - R_{Schw}}$$

$$\frac{d\psi_{eff}}{dr} = 0 = \frac{GM}{(r - R_{Schw})^2} - \frac{l^2}{r^3} \longrightarrow l_K = \sqrt{\frac{GMr^3}{(r - R_{Schw})^2}}$$

For a given radius the angular momentum is specified uniquely, but the reverse is not true: for a given angular momentum we have two solutions instead of one!

The outer orbit is stable, the inner one is unstable.





We have such effect in full GR, in the Schwarzschild metric !

3. Circular orbits in pseudo-Newtonian potential

When the chose angular momentum decreases, the two orbits approach each other, and finally instead of two maxima, only **a single inflection point remains.**
Is inflection point is determined by the conditions

$$\frac{d\psi_{eff}}{dr} = 0$$

and

$$\frac{d^2\psi_{eff}}{dr^2} = 0$$

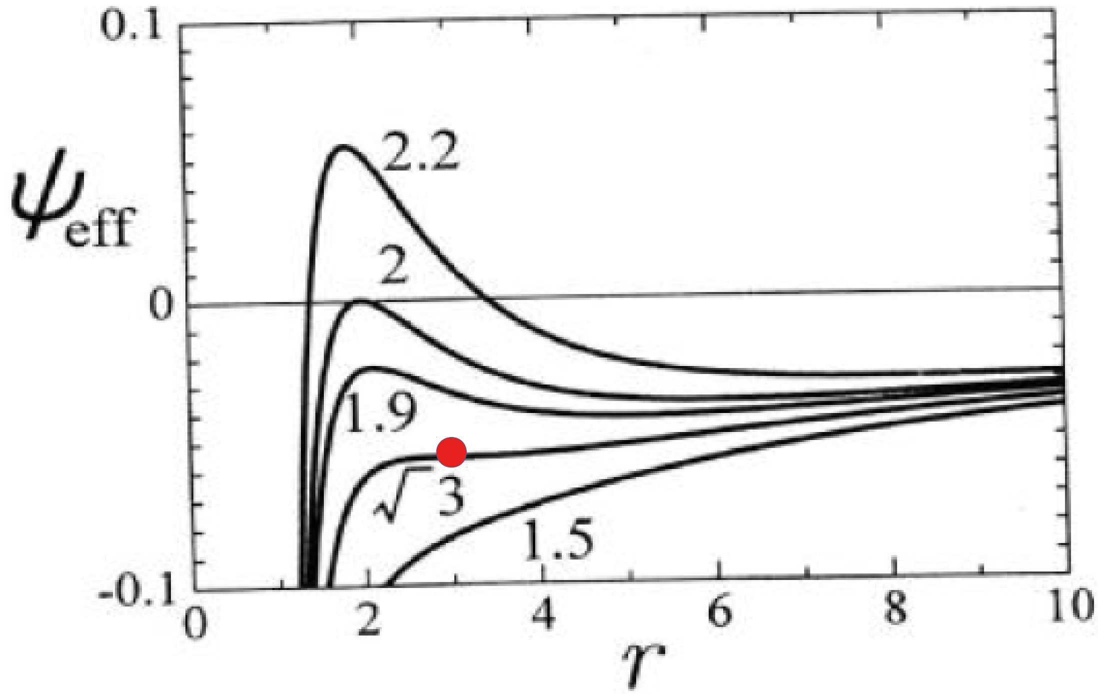


Fig. 2.6 Effective potential $\psi_{\rm eff}$ as a function of r for several values of ℓ. The units of r, $\psi_{\rm eff}$, and ℓ are r_g, c^2, and $r_g c$, respectively, where r_g $(= 2GM/c^2)$ is the Schwarzschild radius. The value of ℓ is attached on each curve.

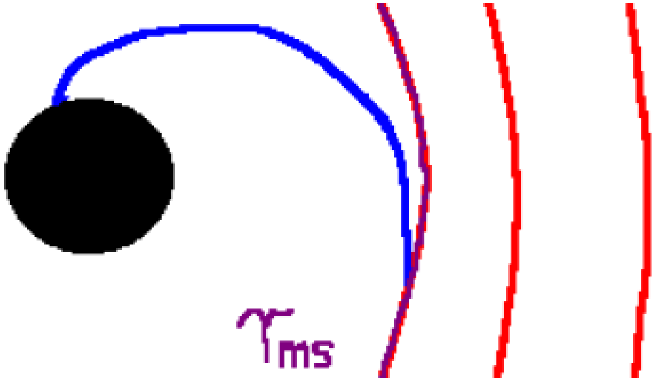
Solving the two equations we obtain both the radius and the corresponding angular momentum. This special orbit is located at

$$\mathbf{r_{ms} = 3\ R_{schw}}$$

and it is known as marginally stable orbit, or more frequently as Innermost Stable Circular Orbit (**ISCO**).

3. Circular orbits in pseudo-Newtonian potential

The existence of ISCO has profound consequences for the accretion process:



- In the case of a Keplerian accretion disk the loss of the angular momentum has to work down to ISCO, further inflow can happen without viscous losses
- The angular momentum there is

$$l_{ms} = \frac{3\sqrt{3}}{2}\sqrt{GMR_{Sch}}$$

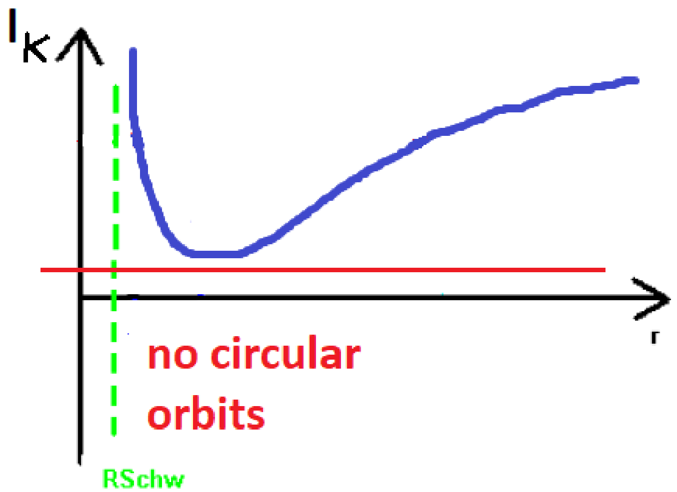
- And the energy there is

$$e_{ms} = -\frac{2GM}{2\,R_{Schw}} + \frac{1}{2}\,\frac{\frac{27}{4}GMR_{Schw}}{(3\,R_{Schw})^2} = -\frac{1}{16}c^2$$

- This sets the efficiency of the disk accretion flow onto a black hole in pseudo-Newtonian potential

η=1/16

- And if the flow is almost Bondi-type but with some angular momentum there is no angular momentum barrier if $l < l_{ms}$.



4. Circular orbits in Schwarzschild metric

In GR full equations of motion have the form

$$\frac{D^2 x^\alpha}{d\tau^2} = 0 = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

and the coefficients are determined from the metric tensor. For Schwarzschild metric we again have conservation laws of energy and angular momentum, so we finally get an equation for the radial velocity

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\frac{e^2}{c^2} + e - \frac{GM}{r} + \frac{1}{2}\frac{l^2}{r^2}\left(1 - \frac{R_{Schw}}{r}\right)$$

And again we can introduce the effective potential which this time have a form

$$\psi_{eff}^{Schwarzschild} = -\frac{GM}{r} + \frac{1}{2}\frac{l^2}{r^2}\left(1 - \frac{R_{Schw}}{r}\right)$$

It is different – it looks like the gravity is not getting stronger as we approach the black hole but the angular momentum barrier is lower. The net effect is the same. Calculating the ISCO we obtain

$$\mathbf{r_{ms} = 3\ R_{schw}}$$

However, the expressions for the angular momentum and the energy at ISCO are slightly different:

$$\boxed{\eta^{Schwarzschild} = 1 - \sqrt{\frac{8}{9}} \approx 0.057}$$

5. Kerr metric

In GR the rotation of the point-like source plays a role. The solution for a point-like source of mass M and dimensionles angular momentum density a has only axial symetry

$$ds^2 = e^{2\nu} dt^2 - e^{2\psi}(d\phi^2 - \omega\, dt)^2 - e^{2\lambda} dr^2 - e^{2\mu} d\theta^2$$

And those nasty functions are:

$$e^{2\nu} = \Delta A / B \qquad A = r^2 + a^2 \cos^2\theta$$

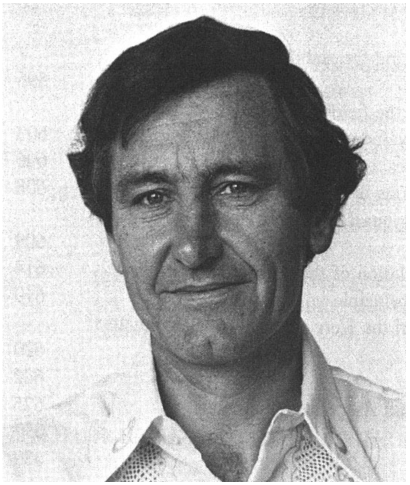
$$e^{2\psi} = B^2 \sin^2\theta / A \qquad B = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta$$

$$\omega = 2 a M r / B \qquad \Delta = r^2 - 2 M r + a^2$$

$$e^{2\lambda} = A / \Delta \qquad -1 < a < 1$$

$$e^{2\mu} = A$$

a=0 Schwarzschild metric



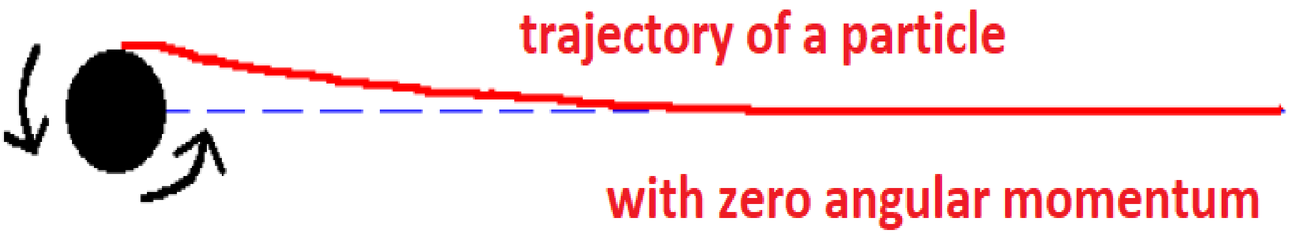
Roy Kerr
Born 1934
Found his solution in 1963

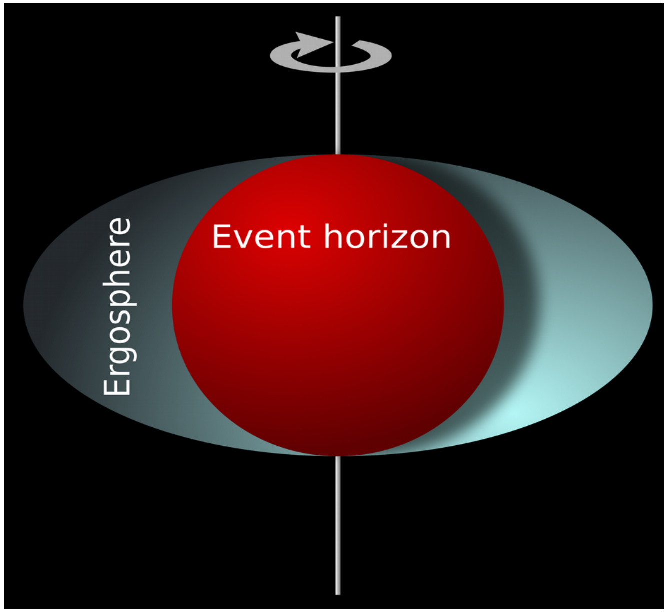
This solution represents a rotating black hole but it does not represent an external field of a rotating star.

5. Kerr metric

In GR the rotation of the point-like source plays a role. The solution for a point-like source of mass M and dimensionles angular momentum density a has only axial symetry.

In ergosphere only a rotational motion consistent with the direction of black hole rotation is possible.





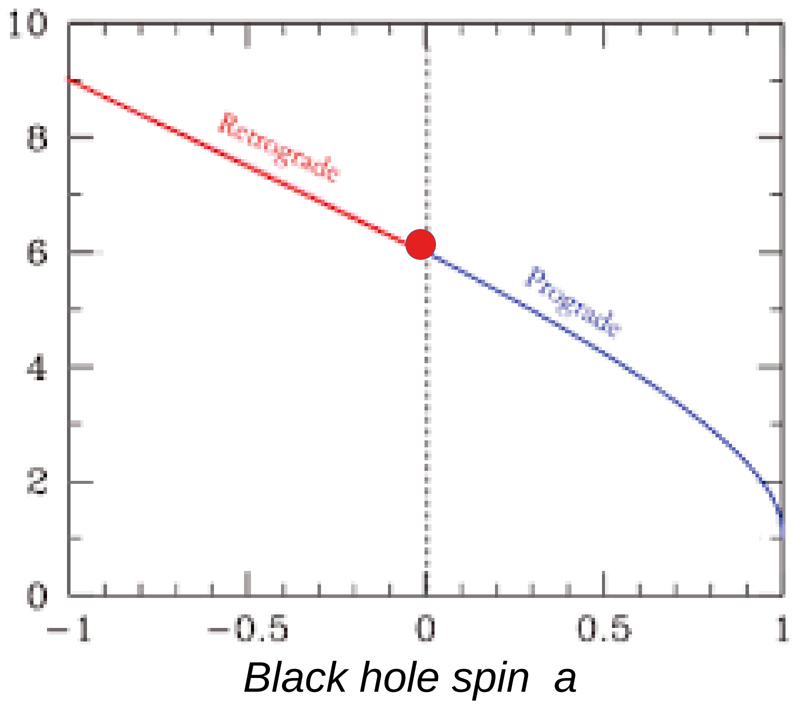
Rotation of a point-like source drags particles in the same direction.

The equation of motion even in the equatorial plane are much more complex:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{r(r^2+a^2)+2a^2M}{r^3}(e+1)^2 - \frac{4aM(e+1)l}{r^3} - \frac{(r-2M)l^2+\Delta r}{r^3}$$

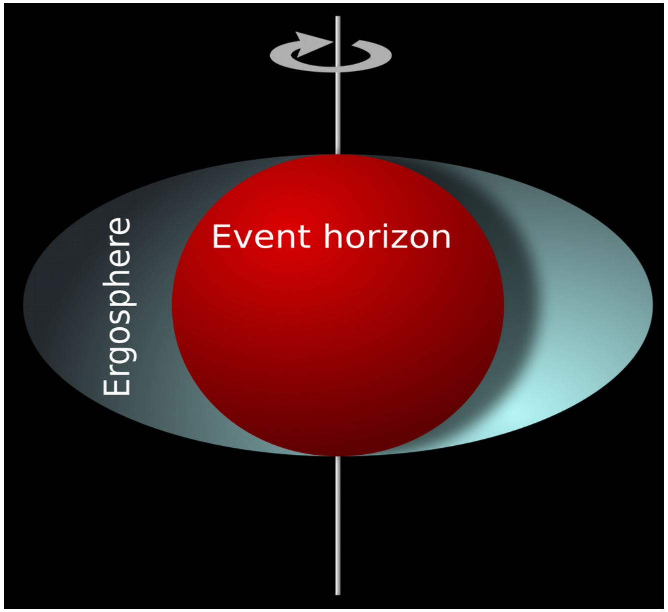
5. Kerr metric

In GR the rotation of the point-like source plays a role. The solution for a point-like source of mass M and dimensionles angular momentum density a has only axial symetry.



Position of ISCO depends on the parameter a, the same with accretion efficiency; here units are

$R_g = 0.5\ R_{Schw}$



From de Rosa et al. (2019)

For maximally rotating black hole realistic maximum angular momentum is a = 0.998, and the corresponding efficiency is

$$\boxed{\eta_{a=0.998} = 0.42}$$

Thus accretion onto a black hole can be an extremely efficient process.

HOMEWORK

- Calculate the angular momentum on circular orbits in Schwarzschild metric. Discuss the problem of $r = 1.5\ r_{Schw}$. How this relates to a black hole shadow/silhuette?
- What is the Newtonian efficiency of accretion if we borrow a concept of ISCO from pseudo-Newtonian approach?