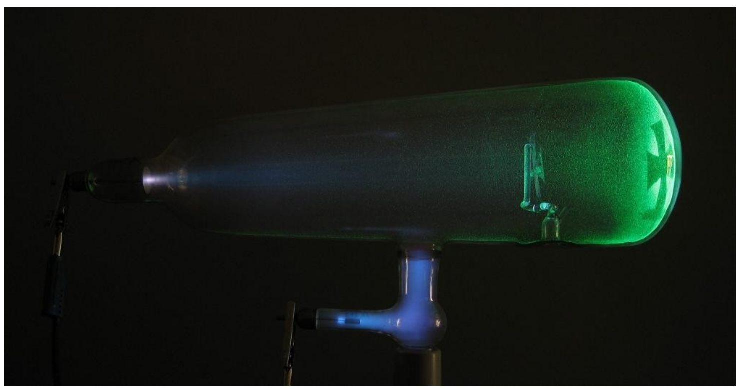
“HIGH ENERGY ASTROPHYSICS” (A subjective introduction)

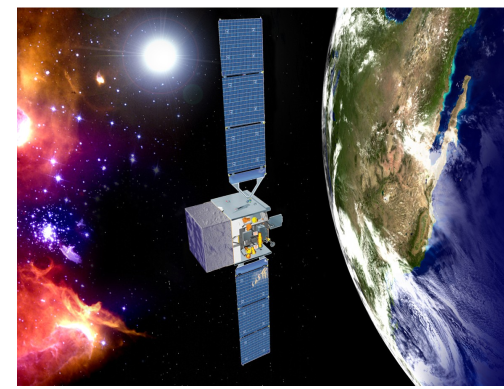
II. Special relativity - reminder

Rafał Moderski
Nicolaus Copernicus Astronomical Center

Story so far ...

High energy astrophysics: *" … areas of research undertaken by astrophysicists, physicists and astronomers interested in high energy phenomena of all types"*



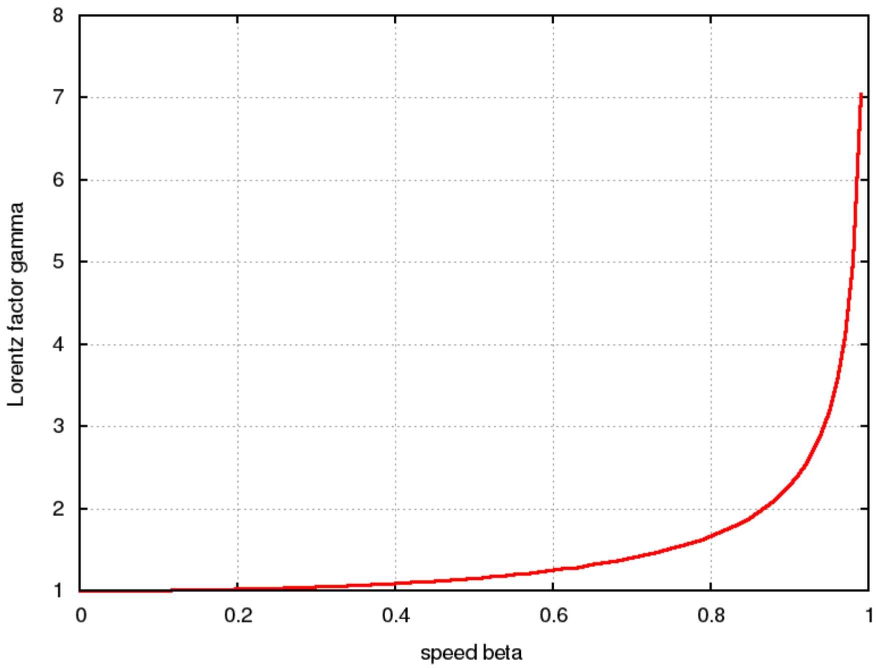


gamma-ray band: $\nu >> 3 \times 10^{19}$ Hz; $E >> 100$ keV

Speed, energy, Lorentz factor

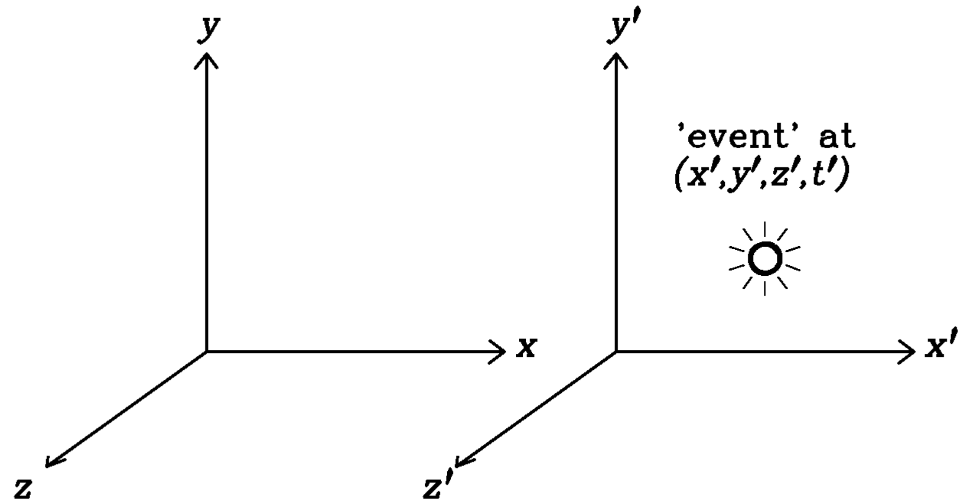
$$\beta = \frac{v}{c}, \qquad \text{Lorentz factor } \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

$$\gamma = 1 \quad \text{for} \quad \beta = 0 \qquad \gamma = \infty \quad \text{for} \quad \beta \to 1$$



β	γ
0.87	2
0.995	10
0.9999995	1000
...	10^7
...	10^{11}

Lorentz transformation



space-time event in K – (ct,x,y,z), in K' – (ct',x',y',z')

Lorentz transformation

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z$$

inverse Lorentz transformation

$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta ct'), \quad y = y', \quad z = z'$$

Length contraction and time dilation

- *length contraction*

$$L' = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma L$$
$$t'_2 = t'_1 \quad \text{and} \quad t_2 = t_1$$

- *time dilation*

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T'$$
$$x'_2 = x'_1$$

Proper time

proper time τ

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

$$d\tau' = ?$$

proper time is a *Lorentz invariant*

Transformation of velocities

for velocity components u_x, u_y, u_z

$$u_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + \beta c\, dt'\right)}{\gamma\left(dt' + \beta\, dx'/c\right)} = \frac{dx'/dt' + \beta c}{1 + \beta/c\, dx'/dt'} = \frac{u'_x + \beta c}{1 + \beta/c\, u'_x}$$

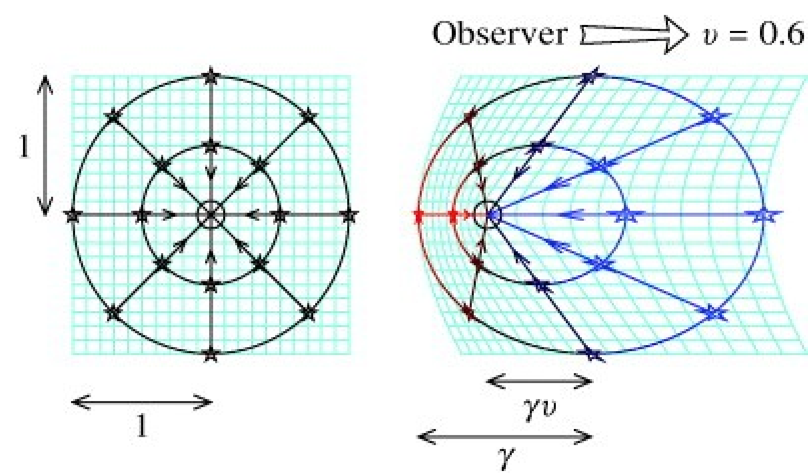
$$u_y = \frac{u'_y}{\gamma\left(1 + \beta/c\, u'_x\right)}, \quad u_z = \frac{u'_z}{\gamma\left(1 + \beta/c\, u'_x\right)}$$

for perpendicular and parallel components

$$u_\parallel = \frac{u'_\parallel + \beta c}{1 + \beta/c\, u'_\parallel}, \quad u_\perp = \frac{u_\perp}{1 + \beta/c\, u'_\parallel}$$

relativistic aberration

$$\tan\theta = \frac{u_\perp}{u_\parallel} = \frac{u' \sin\theta'}{\gamma\left(u' \cos\theta' + \beta c\right)}$$



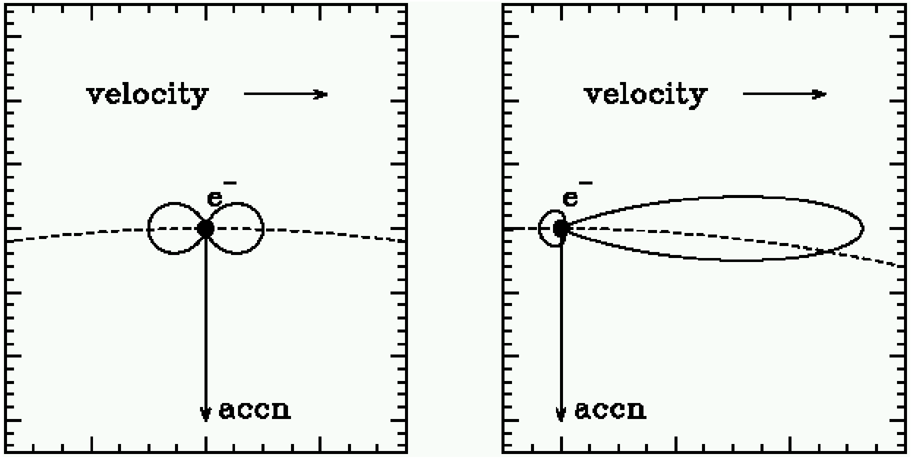
Relativistic beaming

Consider a photon (u'=c) emitted perpendicularly (Θ'=π/2) to the direction of motion:

$$\tan\theta = \frac{c\sin\pi/2}{\gamma\left(c\cos\pi/2+\beta c\right)} = \frac{1}{\beta\gamma}$$

for large velocities

$$\theta \approx \tan\theta \approx \frac{1}{\gamma}$$



Doppler effect

non-relativistic case $\omega_{obs} = \dfrac{1}{(1-\beta\cos\theta)}\,\omega_{source}$

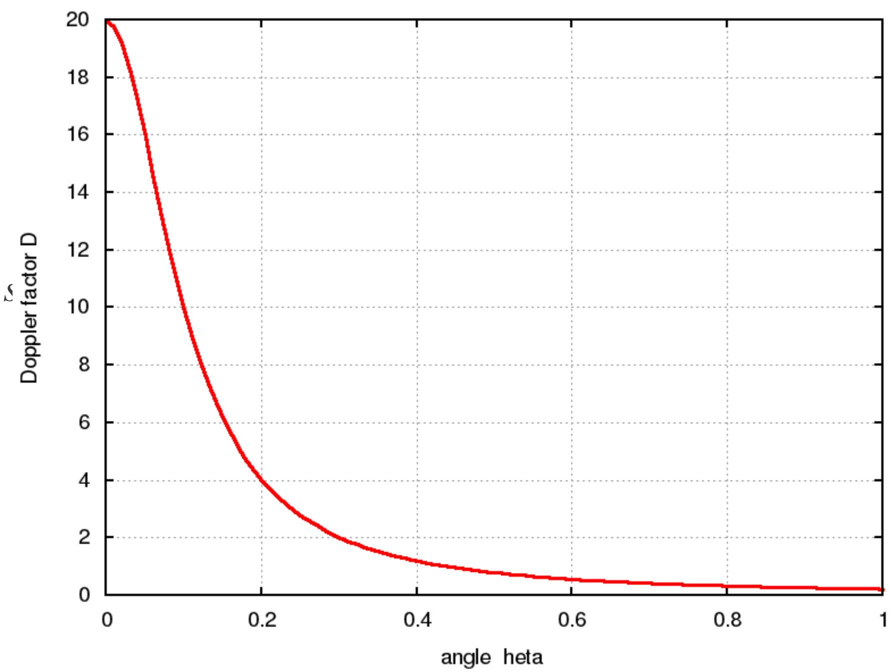
relativistic case

$$\omega_{source} = \omega' = \frac{2\pi}{\Delta T'}$$

$$\Delta T = \gamma\,\Delta T' = \gamma\,\frac{2\pi}{\omega'}$$

$$\Delta T_{obs} = \Delta T - \frac{d}{c} = \Delta T\,(1-\beta\cos\theta)$$

$$\omega_{obs} = \frac{2\pi}{\Delta T_{obs}} = \frac{1}{\gamma\,(1-\beta\cos\theta)}\,\omega_{source} = D\,\omega_{s}$$



Doppler factor D

$$D = \frac{1}{\gamma\,(1-\beta\cos\theta)}$$

$$\theta = 0 \;\Rightarrow\; D = 2\gamma\,,\ \theta = 1/\gamma \;\Rightarrow\; D = \gamma\,,\ \theta = \pi/2 \;\Rightarrow\; D = 1/\gamma$$

Tensor calculus

vector

$$\vec{x}=(x^{1,}x^{2,}x^{3})=x^{1}\hat{e}_{1}+x^{2}\hat{e}_{2}+x^{3}\hat{e}_{3}=x^{i}\hat{e}_{i}$$

Einstein summation convention

$$a_{i}b^{i}=\sum_{i=1}^{i=3}a_{i}b^{i}$$

scalar product

$$\vec{x}\circ\vec{y}=(x^{i}\hat{e}_{i})\circ(y^{k}\hat{e}_{k})=(\hat{e}_{i}\circ\hat{e}_{j})x^{i}y^{j}=g_{ij}x^{i}y^{j}$$

metric tensor

$$g=(g_{ij})=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Tensor calculus

four-vectors

$$X^{\mu}=(x^{0},x^{1},x^{2},x^{3})=x^{0}\hat{e}_{0}+x^{1}\hat{e}_{1}+x^{2}\hat{e}_{2}+x^{3}\hat{e}_{3}=x^{\mu}\hat{e}_{\mu}$$

convention

$$i,j,k,\ldots \text{run } 1,2,3 \qquad \mu,\nu,\xi,\ldots \text{run } 0,1,2,3$$

special relativity metric tensor

$$\eta=(\eta_{ij})=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

scalar product

$$c^{2}d\tau=-\eta_{\mu\nu}dx^{\mu}dx^{\nu}\equiv -ds^{2}$$

Tensor calculus

contra- and *covariant coordinates*

$$X_\alpha = g_{\alpha\beta} X^\beta, \quad X^\alpha = g^{\alpha\beta} X_\beta, \quad g^{\alpha\lambda} g_{\lambda\beta} = \delta^\alpha_\beta$$

contravariant vector transformation

$$x'^\mu = \Lambda^\mu_\nu x^\nu, \quad \Lambda^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}$$

Lorentz transformation

$$(x'^\mu) = \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$dx'^\mu = \Lambda^\mu_\nu dx^\nu$$

covariant vector transformation

$$x'_\mu = \tilde{\Lambda}^\nu_\mu x_\nu, \quad \tilde{\Lambda}^\xi_\mu \Lambda^\nu_\xi = \delta^\nu_\mu$$

Tensor calculus

Lorentz invariance

$$A'_{\mu} B'^{\mu} = \left(\tilde{\Lambda}^{\xi}_{\mu} A_{\xi}\right)\left(\Lambda^{\mu}_{\lambda} B^{\lambda}\right) = \tilde{\Lambda}^{\xi}_{\mu} \Lambda^{\mu}_{\lambda} A_{\xi} B^{\lambda} = \delta^{\xi}_{\lambda} A_{\xi} B^{\lambda} = A_{\lambda} B^{\lambda} = A_{\mu} B^{\mu}$$

Example

$$dx^{\mu} dx_{\mu} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = -c^2 d\tau = ds^2$$

Four-vectors

four-velocity

$$(U^{\mu}) \equiv \frac{dx^{\mu}}{d\tau} = \gamma \frac{dx^{\mu}}{dt} = \gamma \left(\frac{dct}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \gamma (c, \vec{v})$$

$$U_{\mu} U^{\mu} = \frac{dx_{\mu}}{d\tau} \frac{dx^{\mu}}{d\tau} = \frac{-c^2 d\tau^2}{d\tau^2} = -c^2$$

four-momentum

$$(p^{\mu}) \equiv m_0 (U^{\mu}) = (\gamma m_0 c, \gamma m_0 \vec{v}) = \left(\frac{\gamma m_0 c^2}{c}, \gamma m_0 \vec{v} \right) = \left(\frac{E}{c}, \vec{p} \right)$$

$$p_{\mu} p^{\mu} = \eta_{\mu\nu} p^{\nu} p^{\mu} = \frac{-E^2}{c^2} + p^2 = -m_0^2 c^2$$

for photon

$$\vec{p} = \frac{E}{c} \hat{e}_p, \qquad p_{\mu} p^{\mu} = \frac{-E^2}{c^2} + p^2 = 0$$

four-acceleration and *four-force*

$$a^{\mu} = \frac{dU^{\mu}}{d\tau}, \qquad F^{\mu} = m_0 a^{\mu} = \frac{dp^{\mu}}{d\tau}.$$

Relativistic electrodynamics

four-current and four-potential

$$(j^{\mu})=(\rho c, \vec{j}), \qquad (A^{\mu})=(\Phi, \vec{A})$$

electromagnetic field tensor

$$F_{\mu\nu}\equiv A_{\nu,\mu}-A_{\mu,\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

Maxwell equations

$$\partial^{\mu} F_{\nu\mu}=\frac{4\pi}{c} j^{\nu}$$

Exercises

1. Compton scattering

Consider a photon scattered of an electron. Use four-momentum conservation law to show that the shift in a photon wavelength Δλ is

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c^2}\left(1 - \cos\theta\right) \quad ,$$

where cos Θ is a scattering angle.

2. Can an isolated electron absorb a photon?