Disk vertical structure and radiative transfer

1. Summary of the previous lecture

We showed that if the accretion disk is (I) stationary (ii) Keplerian (iii) optically thick (iv) radiates locally as a black body we can uniquely determine:

The radiation flux emitted by the disk as the function of radius

$$F_{rad}(r) = \frac{3GM\dot{M}}{8\pi r^3}\left(1-\frac{l_{inn}}{l_K(r)}\right)$$

The effective temperature as a function of radius

$$\sigma T_{eff}^4 = F_{rad}$$

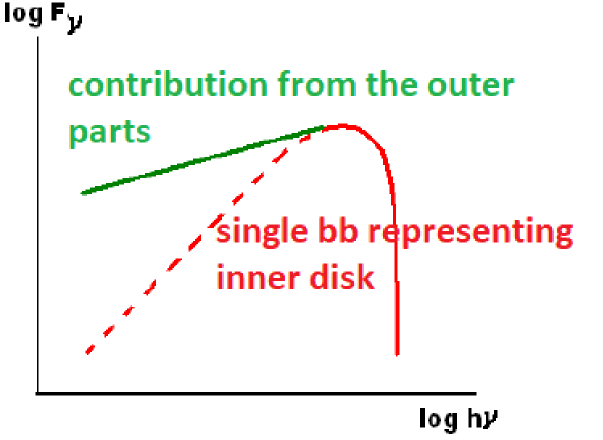
The radiation spectrum of the disk

$$F_\nu \propto \nu^{1/3}\frac{\cos i}{D^2}(M\dot{M})^{2/3} f(\nu), where\, f(\nu)\simeq 1\, for\, small\, \nu$$

HOWEVER:

This approach does not give us any information about

- Disk geometrical and optical thickness
- Disk temperature close to the equatorial plane
- Disk local density
- Stability of the solution



2. alpha – viscosity

Previously we used the angular momentum conservation and energy conservation to get rid of the information about the viscosity. But if we want to get all information about the disk, we need some viscosity prescription.

Shakura (1972) and Shakura&Sunyaev (1973) introduced the following new idea for the viscous stress

$$t_{r\phi} = \alpha P_{tot} \qquad P_{tot} = P_{gas} + P_{rad}$$

The motivation comes from different arguments:

$$\Theta = \int_{-\infty}^{\infty} 2\pi r^2 t_{r\phi}\, dz$$

$$\dot{M}\left(l_K - l_{inn}\right) = \Theta - \Theta_{inn}$$

- Dimensionally, the viscous torque has the dimension of the pressure
- The phenomenon is likely magnetic in nature, as argued in SS73, and the strength of the magnetic field in the disk should roughly scale also with the pressure
- If the viscosity would be turbulent in nature, then this scaling is also justified but to see that we need an equation of hydrostatic equilibrium (coming soon)

We now know that disk viscosity is related to magneto-rotational instability, but this will be discussed during the lecture 11

2. alpha – viscosity

The issue of the hydrodynamics and magnetohydrodynamics of the flow between the two rotating cyliders is studies theoretically as well as in the laboratories.

The replacement of all these complex phenomena with just a simple prescription is a gross oversimplification with not well understood consequences.

But on the other hand it allows to calculate simple disk models efficiently, and then by comparison to the data we can try to judge if the description is reasonable or not that much.

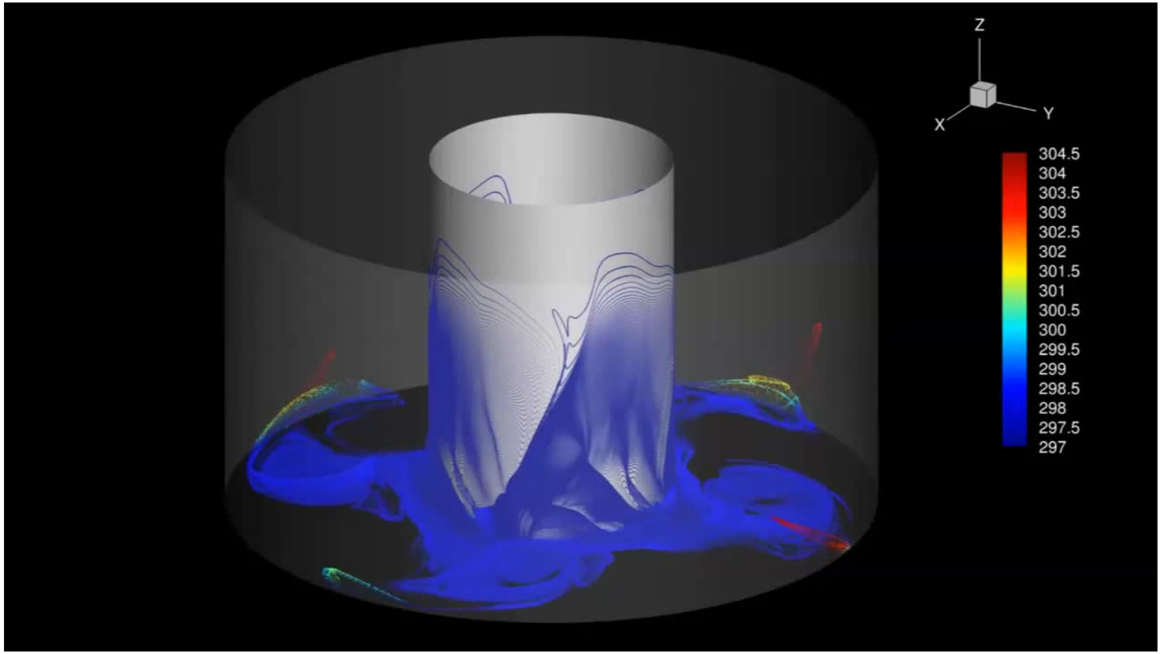


Figure from von Larcher et al. (2018)

The representation of all complex instabilities with just a single expression

$$t_{r\phi} = \alpha P$$

relies on assumption that those instabilities finally saturate at some level of the turbulence, and they transport angular momentum at the requested steady rate.

3. hydrostatic equation

We did not look yet at the ‘z’ component of the Euler equation. In general, in cylidrical coordinates, it reads (where theta stands for phi in usual notation)

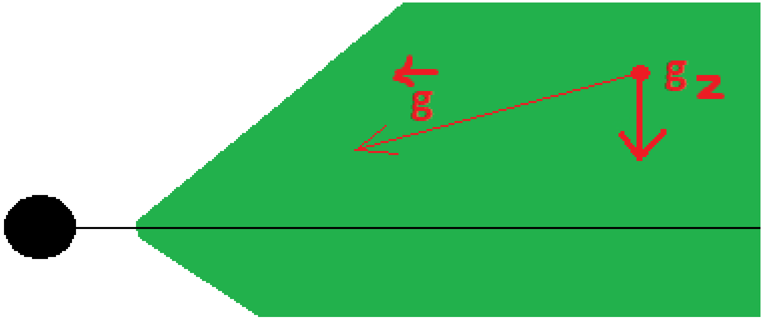
$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\left(\frac{1}{r}\frac{\partial(r\tilde{\tau}_{rz})}{\partial r}+\frac{1}{r}\frac{\partial\tilde{\tau}_{\theta z}}{\partial \theta}+\frac{\partial\tilde{\tau}_{zz}}{\partial z}\right)+\rho g_z$$

But we assume now stationarity and axial symmetry, and all torque components apart from $t_{r\varphi}$ vanish. In addition, we assume that v_z is actually negligible since in general it is much smaller that radial velocity. Thus the equation reduces to the right hand side.

Here g_z represents the ‘z’ component of gravity, where the total gravitational acceleration directed towards the black hole is $g=\frac{GM}{r^2+z^2}$ with the z-component: $g=\frac{GM}{r^2+z^2}\frac{z}{(r^2+z^2)^{1/2}}$

Now assuming z << r we can neglect z in denominator

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3}$$ *hydrostatic equilibrium*



3. hydrostatic equation

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3} \qquad \textit{hydrostatic equilibrium}$$

If we assume politropic approximation $P\rho^{-\gamma} = const = K$ we can solve this equation analytically and we get

$$\frac{K\gamma}{\gamma-1}\rho^{\gamma-1} = -\frac{1}{2}\frac{GM}{r^3}z^2 + const$$

Constant value can be conveniently expressed noticing that the density at the disk surface z = H is equal zero, and when z = 0, the density is given by the density in the equatorial plane, ρe. Since we do not know the constant K, it is convenient to rearrange the expression using again both pressure and density, and then from the boundary conditions we get

$$\frac{P_e}{\rho_e} = \frac{\gamma-1}{2\gamma}\Omega_K^2 H^2$$

This is interesting, remembering the definition of the sound speed and ignoring the numerical factors we see that in the equatorial plane

$$\frac{v_s}{v_\phi} = \frac{H}{r}$$

i.e. sound speed is much smaller than the Keplerian velocity. I remind you that our approach actually required H << r. This is true independently from the politropic approximation.

3. hydrostatic equation

In disk modeling we do not use the universal politropic approximation for the reason which will be clear soon (pressure and opacity issue). And to derive simple vertically integrated/averaged equations we can use even simple approach:

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3} \qquad \textit{hydrostatic equilibrium}$$

$$\frac{P}{\rho H} = \Omega_K^2 H$$

Saves time, and coefficients are not very accurate in such approach anyway...

4. radial force equation

We do not need it, but let us have a look now why we do not need it.

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial(r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z}\right) + \rho g_r$$

From the assumption of stationarity, axial symmetry, and the dominance of the component $t_{r\varphi}$ most components vanish. Leading terms: gravity and centrifugal force. We also usually assume that the radial velocity does not depend on the parameter z, which removes the last term on the LHS.

4. radial force equation

$$v_r \frac{dv_r}{dr} + \frac{1}{\rho}\frac{dP}{dr} = \left(\Omega^2 - \Omega_K^2\right) r$$

Can we safely assume that $l = l_K$ and forget about this equation?

The second term on the LHS is of order of P/ρ/r, and from the equation of the hydrostatic equilibrium we have

$$\frac{1}{\rho}\frac{dP}{dr} \approx \frac{P}{\rho\, r} \approx \Omega_K^2\, r \left(\frac{H}{r}\right)^2$$

The radial pressure gradient is small if the disk in geometrically thin, H<<r.

This is true if the Eddington ratio of the disk is small, we will see that when we use other equations later on. The first term is small if the radial velocity is smaller than the sound speed. This is not true very close to ISCO, independently from the Eddington ratio.

Condition for the Keplerian disk: H << r

5. energy conservation

We talked about this equation during the previous lecture. Under assumptions we introduce to describe the Keplerian disk it reads

$$0 = -\frac{dF_{rad}}{dz} - t_{r\phi} r \frac{d\Omega}{dr}$$

Since now we have an explicit assumption $t_{r\phi} = \alpha P_{tot} \qquad P_{tot} = P_{gas} + P_{rad}$ we can put there the derivative of the Keplerian angular momentum and we get the equation

$$\frac{dF_{rad}(z)}{dz} = \frac{3}{2} \alpha P \Omega_K \qquad \textit{energy generation}$$

This does not close our set of equations even if we supplement it with the expression for the relation between the pressure and the temperature and density

$$P = \frac{k}{\mu m_H} \rho T + \frac{1}{3} a T^4$$

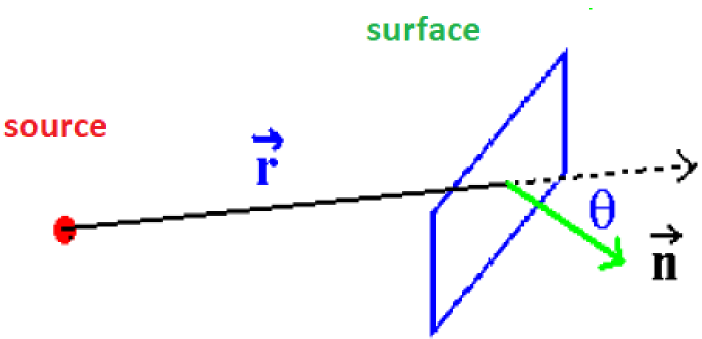
The first term is the gas pressure, and the second term is the radiation pressure in the optically thick medium. Now we have two equations for the vertical structure (for P(z) and $F_{rad}(z)$, continuity equation will not help since it contains new variable v_r.) We need to describe disk cooling, since this will determine the temperature profile of the disk in the vertical direction.

6. Radiative transfer - a short summary

Transfer of radiation from the source through the medum to the observer allows to connect the information about the source and the medium.

(a) general approach

The basic quantity in the radiative transfer theory is the radiation intensity: $I_\nu(\vec{r}, \vec{n}, t)$



This is defined as the energy flux DE per unit surface, per steradian, per unit time, and per unit frequency in the direction n

dE = I_ν A cosθ dν dθ dt

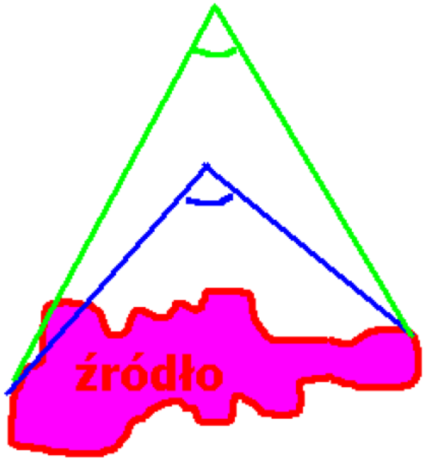
dI_ν does not depend on the distance from the source if there are no losses (absorption or scattering) or gains (emission from the medium between the source and the surface).

The radiation flux in turn is defined as the energy flu dE per unit area, unit time and unit frequency,

$$F_\nu = \int I_\nu \cos\theta \, d\Omega$$

and this quantity does depend on the distance from the source

$$F_\nu \approx I_\nu \Delta\Omega \approx I_\nu \frac{S}{d^2}$$



6. Radiative transfer - a short summary

$$dE = I_\nu \, dA \cos\theta \, d\nu \, d\Omega \, dt$$

Thus, studying intensity is like a stydying a single light ray. If we are in an isotropic medium, then

$I_\nu \neq 0$ but $F_\nu = 0.$

The basic radiation transfer equation has the following form

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

This part, proportional to I_ν and negative, describes extinction (absorption and scattering), but it also contains stimulated emission so sometimes the term can be positive (laser action)

This part described the emission of a new photon with the frequency ν, and also redirection of photons going previously in a different direction to the studied direction by scattering.

This classical form of the radiation transfer does not include the change of the photon energy during the inelastic scattering (Comptonization).

6. Radiative transfer - a short summary

(b) special case – black body emission

This takes place if

$$I_\nu = B_\nu(T)$$

(c) special case – thermal emission

This case is more general, it describes the emission of the matter which is in thermal equilibrium with black body radiation. If we neglect the scattering then we have

$$\frac{dI_\nu}{ds} = 0 = -\alpha_\nu B_\nu + j_\nu \quad \longrightarrow \quad j_\nu = \alpha_\nu B_\nu$$

Kirchhoff law describes the relation between the emissivity of the matter and absorption.

(d) general case

In general solving the radiative transfer equation is not easy and requires considerable computer power. Source of problems:

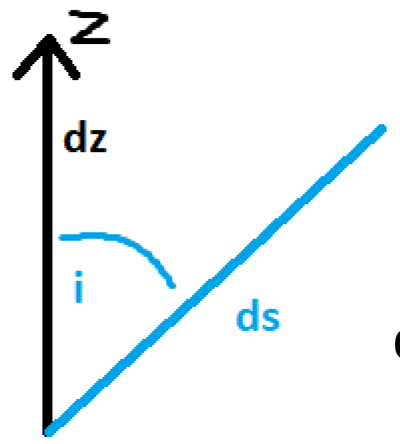
- The equation has to be solved for many directions
- The description of the absorption and emission requires careful independent computation of the ionization structure of the matter, and then computations of the atomic transitions (hundreds of thousands of lines have to be included, on the top of free-free and bound-free transitions)
- Comptonization (inelastic scattering)

6. Radiative transfer - a short summary

(e) special case – optically thick medium

This is also a situation when considerable simplifications can be done. This limit is important for computation the stellar structure as well as the vertical structure of accretion disks.
In the case of a disk, we then concentrate on the vertical direction, since the derivatives will be the largest in this direction

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$



$$ds = dz/\mu$$

$$I_\nu = \frac{j_\nu}{\alpha_\nu} - \frac{\mu}{\alpha_\nu}\frac{\partial I_\nu}{\partial z}$$

Isotropic term

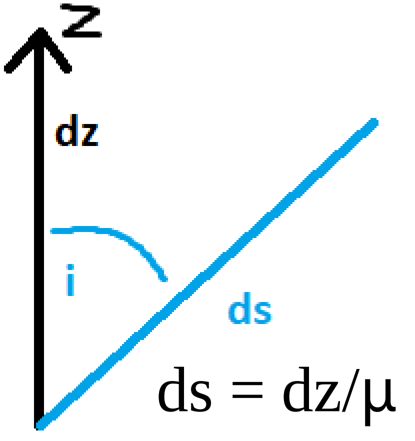
Anisotropic term

Now we construct the radiation flux:
The isotropic term will not contribute since the same amount of radiation passes up and down.

$$F_\nu = 2\pi \int I_\nu \mu \, d\mu$$

6. Radiative transfer - a short summary

$$I_\nu = \cancel{\frac{j_\nu}{\alpha_\nu}} - \frac{\mu}{\alpha_\nu}\frac{\partial I_\nu}{\partial z} \qquad F_\nu = 2\pi \int I_\nu \mu \, d\mu$$



Now we assume that the disk interior radiates as a black body:

$$I_\nu = B_\nu(T)$$

$$\frac{\partial I_\nu}{\partial z} = \frac{\partial B\nu}{\partial z} = \frac{\partial B\nu}{\partial T}\frac{\partial T}{\partial z}$$

and integrating $\mu 2$ over μ from -1 to 1 we get 2/3. The remaining factor are not angle dependent, so we finally obtain an equation

$$F_\nu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}$$

This looks simpler but still the whole equation has to be solved as a function of the vertical coordinate ‘z’ and the frequency, and we need a complex information on opacity coefficient α_ν.

6. Radiative transfer - a short summary

Since we expect that the disk surface will emit as a black body we hope that the detailed dependence of the opacity on the frequency is not that important. We thus integrate this equation over frequency to get the total, frequency-integrated flux as a function of the optical depth

$$F_\nu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z} \quad \Longrightarrow \quad F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}$$

This was possible because we introduced a new quantity – frequency-integrated opacity which is known as the Rosseland mean

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu}$$

Note that the prescription for the Rosseland mean was forced by the structure of radiative transfer equation in the optically thick medium!

The opacity here contains both absorption and electron scattering (*elastic scattering* – not Comptonization!) .
There are analitical formulae for κ_{absor} as functions of density and temperature but available tables are usually much more accurate.

$$\alpha_R = \rho(\kappa_{es} + \kappa_{absor})$$

6. Radiative transfer - a short summary

Electron scattering opacity is directly related to Thomson crosss-section. In cgs opacity has units of cm^2/g (roughly $\kappa_{es} = \sigma_{Th}/m_p$). More accurately

$$K_{es} = 0.20(1 + X) \quad [cm^2/g]$$

where X is the hydrogen abundance (X = 1 for pure hydrogen plasma). In the case of free-free emission of solar-composition plasma

$$K_{ff}(\rho, T) = 0.64 \times 10^{23} \, \rho \, T^{-7/2} \; [cm^2/g]$$

This neglects the ionization and recombination (bound-free transitions) which are important, as well as dust/grain opacity etc. important at lower temperatures.

There is also another frequency-averaged opacity, so called Planck mean, or Planck opacity

$$\kappa_{Pl} = \frac{\int_0^\infty \kappa_\nu B_\nu d\nu}{\int_0^\infty B_\nu d\nu}$$

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\nu} \frac{\partial B_\nu}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} d\nu}$$

It has a different structure and actually applies to optically thin media, in calculating the radiation pressure. For standard disks, we will use Rosseland mean.

6. Radiative transfer - a short summary

We now drop the partial derivative since we assume that the disk is stationary, and we rewrite the equation into the form stressing its differential equation nature

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}$$

$$\frac{dT(z)}{dz} = -\frac{3\alpha_R}{16\sigma T^3}F(z) \qquad \textit{radiative transfer}$$

This is the last equation we need in order to determine the disk vertical structure.

7. Final set of equations for the Keplerian stationary disk structure

We this have three differential equations for the disk vertical structure:

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3} \qquad \textit{hydrostatic equilibrium}$$

$$\frac{dF_{rad}(z)}{dz} = \frac{3}{2}\alpha P \Omega_K \qquad \textit{energy generation}$$

$$\frac{dT(z)}{dz} = -\frac{3\alpha_R}{16\sigma T^3} F_{rad}(z) \qquad \textit{radiative transfer}$$

What happened to the integrated equations we had before? They serve as the boundary conditions. So we need to determine the density ρ, the temperature T, the pressure P, the radiation flux F_{rad} as functions of the coordinate z at each radius. We have 3 equations, but we have also the expression for pressure $P = \frac{k}{\mu m_H}\rho T + \frac{1}{3} a T^4$ and we need to know $\alpha_R(\rho, T)$.

The integration requires boundary conditions. In the equatorial plane $F_{rad} = 0$ (from symmetry!), but we do not know T_e, and ρ_e. On the other hand, at the disk surface (z = H, a priori unknown) we have: ρ(H)=0, F_{rad}(H) is the total flux dissipated in the disk, and we know the effective temperature, T_{eff}.

$$F_{rad}(r) = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \frac{l_{inn}}{l_K(r)}\right)$$

7. Final set of equations for the Keplerian stationary disk vertical structure

Since the boundary conditions are not set at the same plase (some at the disk surface, some at the equatorial plane), the problem is solved eithe by relaxation method (recommended) or by assuming unknown H, calculating the effective temperature and flux at the surface from

$$F_{rad}(r) \quad = \frac{3GM\dot{M}}{8\pi r^3}\left(1 - \frac{l_{inn}}{l_K(r)}\right)$$

and integrating equations down to the equatorial plane.

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3} \qquad \textit{hydrostatic equilibrium}$$

$$\frac{dF_{rad}(z)}{dz} = \frac{3}{2}\alpha P \Omega_K \qquad \textit{energy generation}$$

$$\frac{dT(z)}{dz} = -\frac{3\alpha_R}{16\sigma T^3}F(z) \qquad \textit{radiative transfer}$$

Usually the expected condition $F_{rad} = 0$ is not satisfied at the equatorial plane z = 0, so then another H is assumed, process repeated until convergence.

Two additional issues:

- Actually, the condition for the disk temperature at the disk surface is set by
  $T(H)^4 = 0.5\, T_{eff}^4$ (T_{eff} is the temperature at the optical depth $\tau = 2/3$; stellar atmosphere theory)

- In accretion disk, apart from radiative transfer, also convective transfer is present, and it can be included in the vertical structure computations (e.g. Różańska et al. 1999). We will not discuss this issue.

8. Vertically averaged disk structure

Sometimes we are not that much interested in the details of the vertical structure but we want to know the disk thickness, the surface density, the temperature at the equatorial plane, and the timescales of the disk variability. Then we go to vertically averaged values

$$\frac{1}{\rho}\frac{dP}{dz} = -\frac{GMz}{r^3} \quad \textit{hydrostatic equilibrium} \quad \Rightarrow \quad \frac{1}{\rho}\frac{P}{H} = \frac{GMH}{r^3}$$

$$\frac{dF_{rad}(z)}{dz} = \frac{3}{2}\alpha P \Omega_K \quad \textit{energy generation} \quad \Rightarrow \quad \frac{F_{rad}}{H} = \frac{3}{2}\alpha P \Omega_K$$

$$\frac{dT(z)}{dz} = -\frac{3\alpha_R}{16\sigma T^3} F_{rad}(z) \quad \textit{radiative transfer} \quad \Rightarrow \quad \frac{T}{H} = \frac{3\alpha_R}{16\sigma T^3} F_{rad}$$

The exact coefficients depend on the averaging or integrating method, but the approach is approximate anyway. The last equation can be roughly rewritten in the form

$$F_{rad} = \frac{\sigma T^4}{\tau}$$

where τ is the total optical depth of the disk (equal α_R H)

Thus of course the disk interior is roughly an order of magnitude hotter than the effective temperature.

8. Vertically averaged disk structure

These equations can be solved analytically and have a power law form if

- the pressure has a simple form, that is the first or the second term dominate

$$P=\frac{k}{\mu m_H}\rho T + \frac{1}{3}aT^4$$

- The opacity has a simple form, that is the first or the second term dominate

$$\alpha_R=\rho\left(\kappa_{es}+\kappa_{absor}\right)$$

$$\frac{1}{\rho}\frac{P}{H} = \frac{GMH}{r^3}$$

$$\frac{F_{rad}}{H} = \frac{3}{2}\alpha P \Omega_K$$

$$\frac{T}{H} = \frac{3\alpha_R}{16\sigma T^3}F_{rad}$$

$$F_{rad}(r) = \frac{3GM\dot{M}}{8\pi r^3}\left(1-\frac{l_{inn}}{l_K(r)}\right)$$

These requirements forced Shakura & Sunyaev to adopt three regions of the accretion disk:

- Inner region ($P = P_{rad}$, $\kappa = \kappa_{es}$)
- Middle region ($P = P_{gas}$, $\kappa = \kappa_{es}$)
- Outer region ($P = P_{gas}$, $\kappa = \kappa_{ff}$)

In each of these regions an analytical solution is provided. The boundary condition has the Newtonian form.

8. Vertically averaged disk structure

$$m = \frac{M}{M_\odot}, \quad \dot{m} = \frac{\dot{M}}{\dot{M}_{cr}} = \frac{\dot{M}}{3 \cdot 10^{-8} \frac{M_\odot}{\mathrm{yr}}} \times \left(\frac{M_\odot}{M}\right),$$

SS73 dimensionless units

$$r = \frac{R}{3R_g} = \frac{1}{6}\frac{Rc^2}{GM} = \frac{M_\odot}{M}\frac{R}{9\,\mathrm{km}}.$$

- Inner region ($P = P_{rad}$, $\kappa = \kappa_{es}$)

$$z_0[\mathrm{cm}] = \frac{3}{8\pi}\frac{\sigma_T}{c}\dot{M}(1 - r^{-1/2}) = 3.2 \cdot 10^6 \dot{m}\mathrm{m}\,(1 - z^{-1/2}) \quad (2.8)$$

Disk thickness

This region is important for AGN – disks are radiation-pressure dominated. It is important to note that (apart from the ISCO) the disk thickness is constant, and it is rising with the accretion rate. Since we require the disk to be geometrically thin it is clear that for Eddington ration close to 1 or above the model will not apply. The disk thickness does not depend on viscosity.

$$u_0\left[\frac{\mathrm{g}}{\mathrm{cm}^2}\right] = \frac{64\pi}{9\alpha}\frac{c^2}{\sigma^2}\frac{1}{\omega\dot{M}(1 - r^{-1/2})}$$
$$= 4.6\alpha^{-1}\dot{m}^{-1}r^{3/2}(1 - r^{-1/2})^{-1},$$

Surface density (ρH)

It depends on viscosity, rises with radius but drops with accretion rate. This will cause the stability issue (lecture 9).

8. Vertically averaged disk structure

$$m = \frac{M}{M_\odot}, \quad \dot{m} = \frac{\dot{M}}{\dot{M}_{cr}} = \frac{\dot{M}}{3 \cdot 10^{-8} \frac{M_\odot}{\text{yr}}} \times \left(\frac{M_\odot}{M}\right),$$

SS73 dimensionless units

$$r = \frac{R}{3R_g} = \frac{1}{6} \frac{Rc^2}{GM} = \frac{M_\odot}{M} \frac{R}{9\,\text{km}}.$$

- Middle region ($P = P_{gas}$, $\kappa = \kappa_{es}$)

$$z_0 = 1.2 \cdot 10^4 \alpha^{-1/10} \dot{m}^{1/5} m^{9/10} r^{21/20} (1 - r^{-1/2})^{1/5}$$

Disk thickness

In this region the disk thickness rises with the radius practically linearly.

$$u_0 = 1.7 \cdot 10^5 \alpha^{-4/5} \dot{m}^{3/5} m^{1/5} r^{-3/5} (1 - r^{-1/2})^{3/5}$$

Surface density (ρH)

It depends on viscosity, but now decreases
with radius but rises with accretion rate.

8. Vertically averaged disk structure

$$m = \frac{M}{M_\odot}, \qquad \dot{m} = \frac{\dot{M}}{\dot{M}_{cr}} = \frac{\dot{M}}{3 \cdot 10^{-8} \frac{M_\odot}{\text{yr}}} \times \left(\frac{M_\odot}{M}\right),$$

SS73 dimensionless units

$$r = \frac{R}{3R_g} = \frac{1}{6}\frac{Rc^2}{GM} = \frac{M_\odot}{M}\frac{R}{9\,\text{km}}.$$

- Outer region ($P = P_{gas}$, $\kappa = \kappa_{ff}$)

$$z_0 = 6.1 \cdot 10^3 \alpha^{-1/10} \dot{m}^{3/20} m^{9/10} r^{9/8} (1 - r^{-1/2})^{3/20}$$

Disk thickness

In this region the disk thickness rises with the radius practically linearly.

$$u_0 = 6.1 \cdot 10^5 \alpha^{-4/5} \dot{m}^{7/10} m^{1/5} r^{-3/4} (1 - r^{-1/2})^{7/10}$$

Surface density (ρH)

It depends on viscosity, but again decreases with radius but rises with accretion rate.

8. Vertically averaged disk structure

Frequently used terms inner-middle-outer regions are a bit misleading.
If you use inner region prescription close to ISCO then

$$u_0\left[\frac{\mathrm{g}}{\mathrm{cm}^2}\right] = 4.6\alpha^{-1}\dot{m}^{-1}r^{3/2}(1-r^{-1/2})^{-1}, \quad \Rightarrow \quad u_0 \rightarrow \infty$$

but actually close to ISCO the temperature is low, and the prescription ‘outer’ should be used, so

ISCO | Outer region | Middle region | Inner region | Middle region | Outer region → r

This approach does not solve all the problems at ISCO (radial velocity is infinite) but otherwise it returns reasonable values almost down to ISCO.

Taking the results for the inner region from SS73 we can also show that

$$\frac{P_{rad}}{P_{gas}} = \dot{m}^2 m^{1/4}$$

Radiation pressure becomes larger in comparison to the gas pressure when Eddington ratio rises, and when the black hole mass is large (for AGN).

9. Acccretion disk behaviour close to ISCO

The understanding of the flow close to ISCO was not obious, and it was a topic of my PhD Thesis. The final formulation of the equations is in the paper ***Muchotrzeb & Paczyński (1982)***, where we used pseudo-Newtonian approximation.

In general case it is important to keep more terms in the vertically integrated equation of the radial motion

$$\frac{1}{\varrho}\frac{dP}{dr} - (\Omega^2 - \Omega_k^2) r + v_r \frac{dv_r}{dr} = 0.$$

to allow for the departure from the Keplerian motion. Consequently, also the energy balance has to be modified

$$\dot{M}(l - l_{in})\left(-\frac{d\Omega}{dr}\right) + B_1 \dot{M} T \frac{dS}{dr} - \frac{dF_r}{dr} = 4\pi r F^-,$$

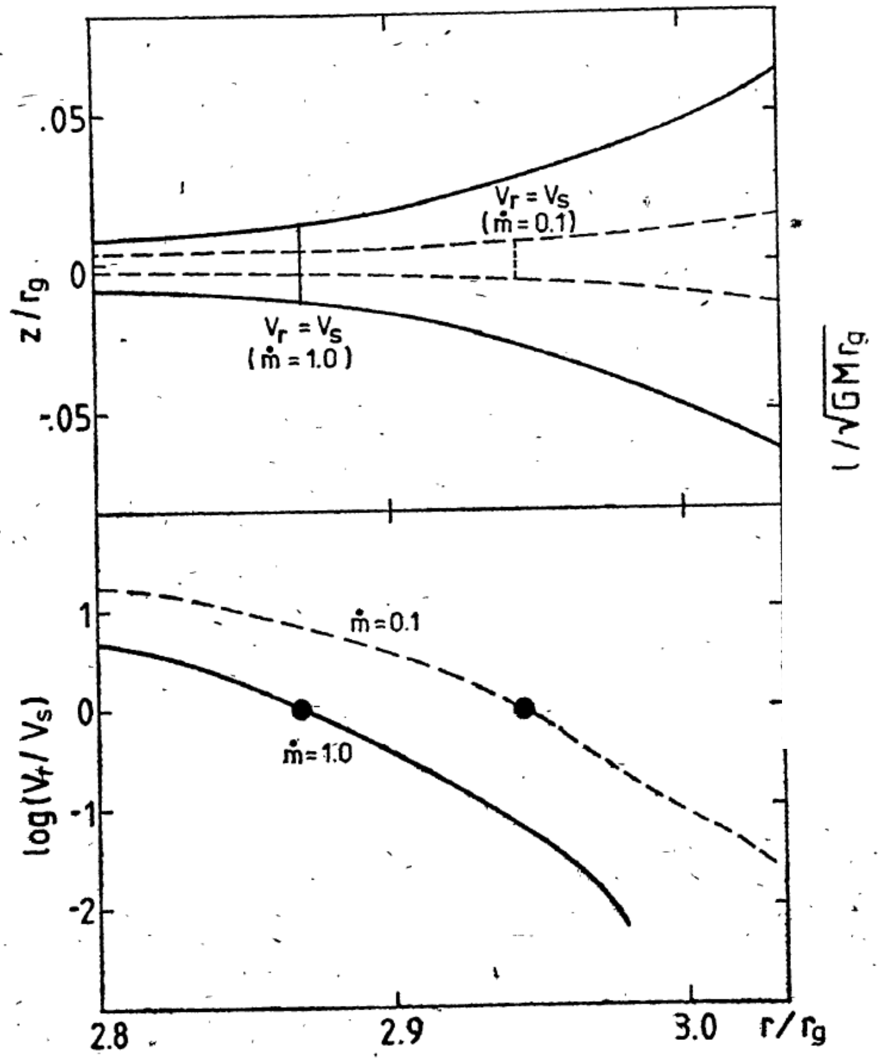
advection term

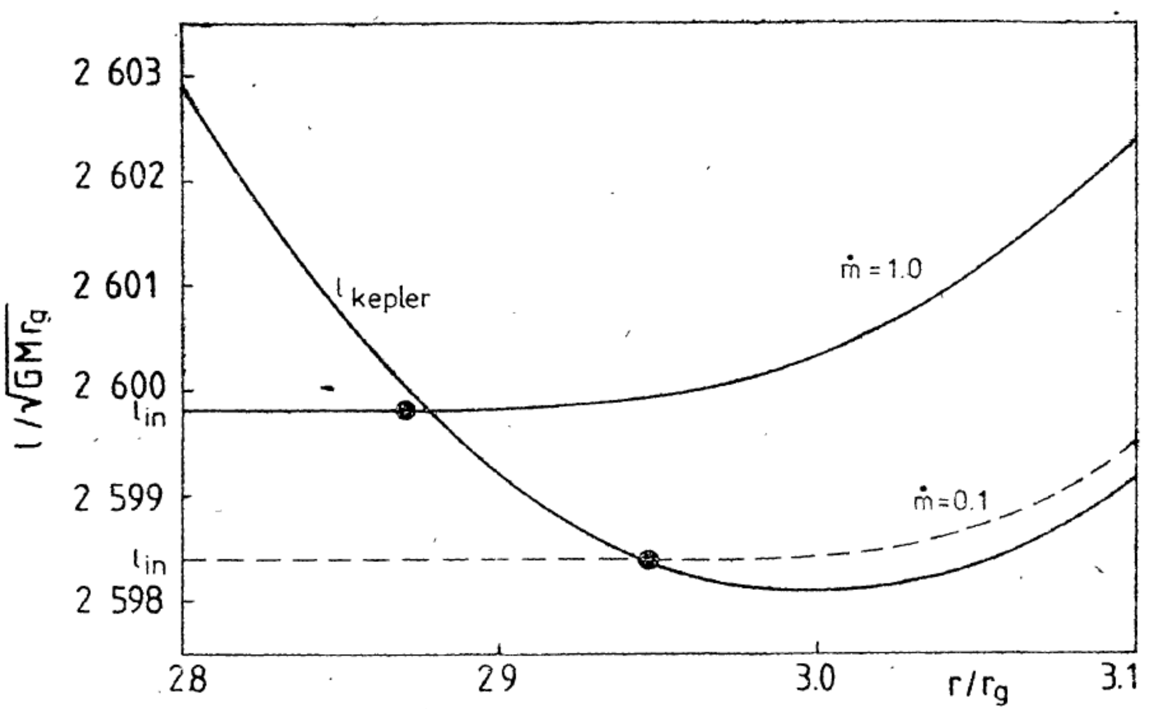
Then the radial motion, combined with the continuity equation can be rewritten in a form

$$\frac{dv_r}{dr} = \frac{a_s^2/r + a_s^2 d\ln z/dr + (\Omega^2 - \Omega_k^2) r}{v_r (1 - a_s^2/v_r^2)},$$

very similar to the form we studied for spherical accretion. Here a_s is a sound speed.

9. Acccretion disk behaviour close to ISCO





solution for the disk structure close to ISCO, here r_g is actually R_{Schw}.

In geometrically thin disk the angular momentum above ISCO is roughly Keplerian, just below ISCO the flow becomes supersonic and angular momentum is constant.

9. Acccretion disk behaviour close to ISCO

Those additional terms become more important in a standard disk as the Eddington ratio of the flow rises, they are also important for optically thin geometrically thick flows. This will be important in lecture 8 and 9. During lecture 9 we will also see why we need this vertical structure.

HOMEWORK

1. How exactly we derived the equation:

$$F(z) = -\frac{16\sigma T^3}{3\alpha_R}\frac{\partial T}{\partial z}$$

Starting from the equation:

$$F_\nu = -\frac{4\pi}{3\alpha_\nu}\frac{\partial B_\nu}{\partial T}\frac{\partial T}{\partial z}$$