“HIGH ENERGY ASTROPHYSICS”
(A subjective introduction)

IV. Radiation processes

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Story so far ...

- various processes in astrophysical fluids: collisions, ionization, viscosity, sound waves
- in some cases shock may form in astrophysical plasma
- Fermi acceleration in shock result in a non-thermal particle distribution with a power-law index ~2

Basic definitions

- *luminosity*

$$dE = L\,dt \quad , \quad dE = L_\nu\,dt\,d\nu \quad , \quad L = \int_0^\infty L_\nu\,d\nu$$

- *energy flux*

$$dE = F\,dA\,dt \quad , \quad dE = F_\nu\,dA\,dt\,d\nu \quad , \quad F = \int_0^\infty F_\nu\,d\nu \quad , \quad F \propto \frac{1}{r^2}$$

- *intensity* (brightness)

$$dE = I_\nu(\Omega)\,dt\,dS\,d\nu\,d\Omega \quad , \quad J_\nu = \frac{1}{4\pi}\int I_\nu(\Omega)\,d\Omega$$

$$F_\nu = \int I_\nu(\Omega)\cos\theta\,d\Omega$$

- *energy density*

$$dE = u_\nu\,dV\,d\nu \quad , \quad u_\nu = \frac{1}{c}\int I_\nu(\Omega)\,d\Omega = \frac{4\pi}{c}J_\nu$$

- *momentum flux* and *radiation pressure*

$$P_\nu = \frac{1}{c}\int I_\nu(\Omega)\cos^2\theta\,d\Omega \quad , \quad P_{\rm iso} = \frac{u}{3}$$

Radiative transfer

- free space

$$\frac{dI_\nu}{ds}=0$$

- emission (*emission coefficient*, j_ν)

$$dE=j_\nu(\Omega)\,dV\,dt\,d\nu\,d\Omega$$

$$dI_\nu=j_\nu\,ds$$

- absorption (*absorption coefficient*, α_ν)

$$dI_\nu=-\alpha_\nu I_\nu\,ds \quad , \quad \alpha_\nu=n\sigma_\nu \quad , \quad \lambda=\frac{1}{\alpha_\nu}=\frac{1}{n\sigma_\nu}$$

- equations of radiative transfer

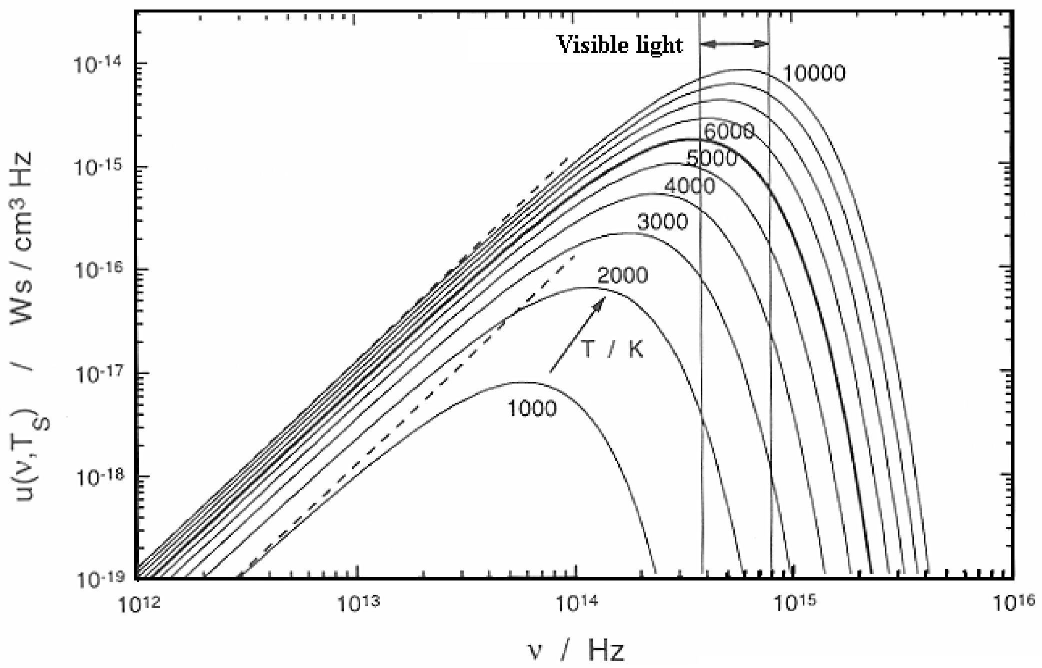
$$\frac{dI_\nu}{ds}=-\alpha_\nu I_\nu+j_\nu$$

pure absorption (j_ν=0)

$$I_\nu(s)=I_\nu(s_0)\exp\left[-\int_{s_0}^{s}\alpha_{(s')}\,ds'\right]=I_\nu(s_0)\,e^{-\tau_\nu}$$

$$\tau_\nu\stackrel{\text{def}}{=}\int_{s_0}^{s}\alpha_{(s')}\,ds' \quad , \quad d\tau_\nu=n\sigma_\nu\,ds=\frac{ds}{\lambda}$$

Blackbody radiation



- Planck law

$$B_\nu(T) = \frac{2h}{c^2} \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

- Stefan-Boltzman law

$$\int_0^\infty B_\nu \, d\nu = \frac{2\pi^4 k_B^4}{15 c^2 h^3} T^4$$

- Rayleigh-Jeans and Wien limits

$$B_\nu(T) = \begin{cases} \frac{2}{c^2} \nu^2 k_B T & \text{for} \quad h\nu \ll k_B T \\ \frac{2h}{c^2} \nu^3 \exp\left(-\frac{h\nu}{k_B T}\right) & \text{for} \quad h\nu \gg k_B T \end{cases}$$

- Wien's displacement law

$$E_{\text{peak}} = h\nu_{\max} \approx 2.82 \; k_B T$$

Temperature definitions

- brightness temperature

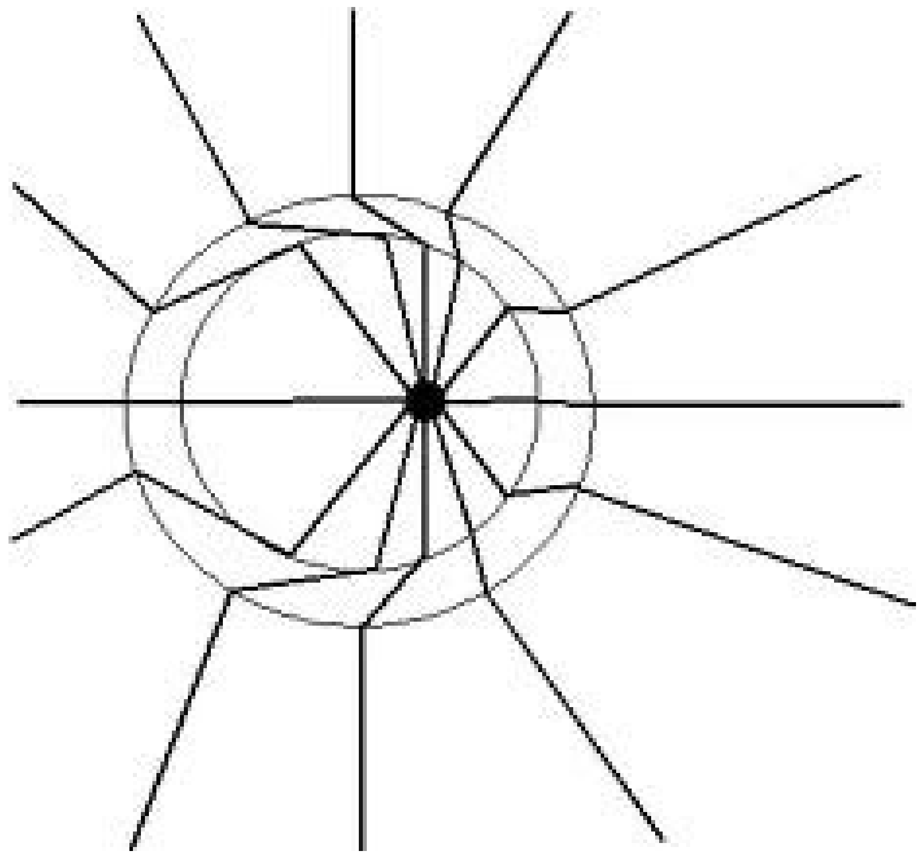
$$I_\nu = B_\nu(T_b)$$

- effective temperature

$$F = \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$$

Radiation from moving charges

- Why do accelerated charges radiate?



Radiation from moving charges

- Maxwell's equations

$$\vec{\nabla} \circ \vec{D} = 4\pi\rho \qquad \vec{\nabla} \circ \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \qquad \vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{D}$$

$$\vec{D} = \epsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \epsilon = \mu = 1$$

- conservation of electric charge

$$\vec{\nabla} \circ (\vec{\nabla} \times \vec{H}) \equiv 0 \quad \Rightarrow \quad \partial_t \rho + \vec{\nabla} \circ \vec{j} = 0$$

- energy conservation (*Poynting's theorem*)

$$\partial_t \left(\frac{E^2 + B^2}{8\pi} \right) + \vec{j} \circ \vec{E} = -\vec{\nabla} \circ \vec{S} \quad ,$$

where *Poynting vector* (*Poynting flux*) [erg s^{-1} cm^{-2}]

$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B}$$

Electromagnetic potentials

- scalar potential Φ and vector potential A

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\partial_t \vec{A}$$

- potentials from charge distribution and current

$$\vec{\nabla}^2 \Phi - \frac{1}{c^2}\partial_t^2 \varphi + \frac{1}{c}\partial_t \left(\vec{\nabla} \circ \vec{A} + \frac{1}{c}\partial_t \Phi \right) = -4\pi\rho$$

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2}\partial_t^2 \vec{A} - \vec{\nabla} \left(\vec{\nabla} \circ \vec{A} + \frac{1}{c}\partial_t \varphi \right) = -\frac{4\pi}{c}\vec{j}$$

- retarded potentials

$$\Phi(\vec{r}, t) = \int \frac{[\rho]\, d^3 r'}{|\vec{r} - \vec{r}\,'|}$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c}\int \frac{[\vec{j}]\, d^3 r'}{|\vec{r} - \vec{r}\,'|}$$

$$[X] \stackrel{\text{def}}{=} X\left(\vec{r}\,', t - |\vec{r} - \vec{r}\,'|/c\right)$$

Liénard-Wiechert potentials

- retarded potentials for single moving charge

$$\vec{u} = \frac{d\,\vec{r}_0(t)}{dt}$$

$$\rho(\vec{r}, t) = q\,\delta[\vec{r} - \vec{r}_0(t)]$$

$$\vec{j}(\vec{r}, t) = q\,\vec{u}\,\delta[\vec{r} - \vec{r}_0(t)]$$

solution

$$\Phi(\vec{r}, t) = \left[\frac{q}{\kappa R}\right] \qquad \vec{A}(\vec{r}, t) = \left[\frac{q\,\vec{u}}{c\,\kappa R}\right]$$

$$t_{\mathrm{ret}} = t - R/c \quad , \quad \vec{R} = R\,\vec{n} = \vec{r} - \vec{r}_0(t_{\mathrm{ret}})$$

$$\kappa = 1 - \frac{1}{c}\vec{n}(t) \circ \vec{u}(t) = 1 - \vec{n} \circ \vec{\beta}$$

Fields from a moving charge

- solution for E and B

$$\vec{E}(\vec{r},t) = q\left[\frac{(\vec{n}-\vec{\beta})(1-\beta^2)}{\kappa^3 R^2}\right] + \frac{q}{c}\left[\frac{\vec{n}}{\kappa^3 R}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)\right]$$

$$\vec{B}(\vec{r},t) = [\vec{n}\times\vec{E}]$$

- velocity field vs. acceleration field

- low velocity without acceleration

$$\beta \rightarrow 0 \quad , \quad \kappa \rightarrow 1 \quad , \quad \dot{\vec{\beta}} = 0$$

$$\vec{E}(\vec{r},t) = q\,\frac{\vec{r}-\vec{r}_0}{(r-r_0)^3}$$

- radiation fields

$$\vec{E}_{\mathrm{rad}}(\vec{r},t) = \frac{q}{c}\left[\frac{\vec{n}}{\kappa^3 R}\times\left((\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)\right] \qquad \vec{B}_{\mathrm{rad}}(\vec{r},t) = [\vec{n}\times\vec{E}_{\mathrm{rad}}(\vec{r},t)]$$

Larmor's formula

- non-relativistic case

$$\beta \ll 1 \quad , \quad \dot{\vec{\beta}} = \frac{\dot{\vec{u}}}{c} \quad , \quad \vec{n} - \vec{\beta} \rightarrow \vec{n} \quad , \quad \kappa^3 \rightarrow 1$$

$$E_{\mathrm{rad}} \sim \frac{q}{c^2} \frac{\dot{u}}{R} \quad , \quad E_{\mathrm{vel}} \sim \frac{q}{R^2} \quad \Rightarrow \quad \frac{E_{\mathrm{rad}}}{E_{\mathrm{vel}}} \sim \frac{\dot{u}\, R}{c^2}$$

near zone R~λ vs. far zone R>>λ

$$\frac{E_{\mathrm{rad}}}{E_{\mathrm{vel}}} \sim \frac{R}{\lambda} \frac{u}{c}$$

- *Larmor's formula* (total emitted power)

$$\frac{dW}{dt\, d\Omega} = \frac{q^2 \dot{u}^2}{4 \pi c^3} \sin^2 \theta$$

$$\frac{dW}{dt} = \frac{2 q^2 \dot{u}^2}{3 c^3}$$

Dipole radiation

- set of non-relativistic particles with size L<<λ

- electric radiation field

$$\vec{E}_{\mathrm{rad}} = \sum_i \frac{q_i}{c^2} \frac{\vec{n} \times (\vec{n} \times \dot{\vec{u}}_i)}{R_i}$$

far away from the source

$$\vec{E}_{\mathrm{rad}} = \sum_i \frac{q_i}{c^2} \frac{\vec{n} \times (\vec{n} \times \dot{\vec{u}}_i)}{R_0} = \frac{1}{c^2 R_0} \vec{n} \times (\vec{n} \times \ddot{\vec{d}})$$

where the *dipole moment* is

$$\vec{d} = \sum_i q_i \vec{r}_i$$

- *dipole approximation*

$$\frac{dW}{dt\, d\Omega} = \frac{\ddot{\vec{d}}^2}{4\pi c^3} \sin^2\theta \quad , \quad \frac{dW}{dt} = \frac{2\,\ddot{\vec{d}}^2}{3 c^3}$$

Spectrum of dipole radiation

- spectrum of radiation dW/dω
- Fourier transform and Parseval's theorem

$$E(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} E(t)e^{i\omega t}dt \quad , \quad E(t)=\int_{-\infty}^{+\infty} E(\omega)e^{-i\omega t}d\omega$$

$$\int_{-\infty}^{+\infty} |E(t)|^2 dt=2\pi\int_{-\infty}^{+\infty} |E(\omega)|^2 d\omega$$

- spectrum

$$S=\frac{dW}{dA\,dt}=\frac{c}{4\pi}|E(t)|^2 \quad \Rightarrow \quad \frac{dW}{d\omega\,dA}=c|E(\omega)|^2$$

- for dipole radiation

$$E(t)=\frac{\ddot{d}(t)}{c^2 R_0}\sin\theta \quad \Rightarrow \quad E(\omega)=-\frac{1}{c^2 R_0}\omega^2 d(\omega)\sin\theta$$

$$\frac{dW}{d\omega\,d\Omega}=\frac{\omega^4}{c^3}|d(\omega)|^2\sin^2\theta \quad , \quad \frac{dW}{d\omega}=\frac{8\pi}{3c^3}\omega^4|d(\omega)|^2$$

Thomson scattering

- scattering of an electromagnetic wave off a free electron

$$\vec{E} = E_0 \sin(\omega t)\hat{e}$$

$$\frac{dW}{dt\,d\Omega} = \frac{q^4 E_0^2}{8\pi m^2 c^3}\sin^2\theta \quad , \quad \frac{dW}{dt} = \frac{q^4 E_0^2}{3 m^2 c^3}$$

- differential cross section

$$\frac{dW}{dt\,d\omega} = \frac{d\sigma}{d\omega}\langle S\rangle$$

$$\langle S\rangle = \frac{c E_0^2}{8\pi}$$

$$\frac{d\sigma}{d\Omega} = \frac{q^4}{m^2 c^4}\sin^2\theta$$

- in case of an electron

$$\frac{d\sigma}{d\Omega} = r_0^2 \sin^2\theta \quad , \quad \sigma_T = \int_\Omega \frac{d\sigma}{d\Omega}\,d\Omega = \frac{8\pi}{3} r_0^2 \quad , \quad r_0 \stackrel{\text{def}}{=} \frac{e^2}{m_e c^2}$$

Bremsstrahlung

- radiation emitted during encounters of charged particles in ionized plasma

- free-free radiation

- in astrophysics mostly radiation from electrons in a fixed Coulomb field of the ions

Bremsstrahlung – single electron

- electron moving past an ion of charge Ze with impact parameter b

$$\vec{d}=-e\vec{r} \quad , \quad \ddot{\vec{d}}=-e\dot{\vec{v}} \quad , \quad -\omega^2 d(\omega)=-\frac{e}{2\pi}\int_{-\infty}^{+\infty}\dot{\vec{v}}\,e^{i\omega t}\,dt$$

$$d(\omega)\sim\begin{cases}\frac{1}{\omega^2}\frac{e}{2\pi}\Delta v & \text{for}\ \omega\ll\omega_c\\ 0 & \text{for}\ \omega\gg\omega_c\end{cases} \quad , \quad \Delta v=\frac{2Ze^2}{mbv}$$

- single particle spectrum

$$\frac{dW(b)}{d\omega}=\begin{cases}\frac{8Z^2e^6}{3\pi c^3 m^2}\frac{1}{b^2v^2} & \text{for}\ \omega\ll\omega_c\\ 0 & \text{for}\ \omega\gg\omega_c\end{cases}$$

- single speed spectrum

$$\frac{dW}{d\omega\, dV\, dt}=\frac{16e^6Z^2}{3c^3m^2}\frac{n_e n_{ion}}{v}\ln\left(\frac{b_2}{b_1}\right) \quad , \quad g_{ff}(v,\omega)=\frac{\sqrt{3}}{\pi}\ln\left(\frac{b_2}{b_1}\right)$$

- *Gaunt factor* g_{ff}

Thermal Bremsstrahlung

- average over thermal distribution of speeds
- for isotropic particle distribution

$$dP \propto v^2 \exp\left(-\frac{m v^2}{2 k_B T}\right) dv$$

$$\frac{dW(T,\omega)}{dV\, dt\, d\omega} = \frac{\int_{v_{min}}^{\infty} \frac{dW(v,\omega)}{d\omega\, dV\, dt} v^2 \exp\left(-\frac{m v^2}{2 k_B T}\right) dv}{\int_0^{\infty} v^2 \exp\left(-\frac{m v^2}{2 k T}\right) dv}$$

- free-free emission of a thermal plasma

$$\frac{dW}{dV\, dt\, d\nu} = \frac{2^5 \pi e^6}{3 m c^3} \left(\frac{2\pi}{3 k_B m}\right)^{1/2} Z^2 n_e n_{ion} T^{-1/2} e^{-h\nu / k_B T} \bar{g}_\nu^{ff}$$

$$\frac{dW}{dV\, dt} = \frac{2^5 \pi e^6}{3 h m c^3} \left(\frac{2\pi k_B}{3 m}\right)^{1/2} Z^2 n_e n_{ion} T^{-1/2} \bar{g}^{ff}$$

- absorption $\alpha^{ff} \sim \nu^{-2}$

Synchrotron radiation

- radiation emitted by a relativistic charge in magnetic field

- charge in magnetic field

$$\frac{dp_\alpha}{d\tau} = \frac{q}{c} F_{\alpha\beta} U^\beta$$

$$\frac{d\left(\gamma m c^2\right)}{dt} = q \vec{v} \circ \vec{E} = 0$$

$$\frac{d\vec{p}}{dt} = \frac{d\left(\gamma m \vec{v}\right)}{dt} = q \frac{\vec{v}}{c} \times \vec{B}$$

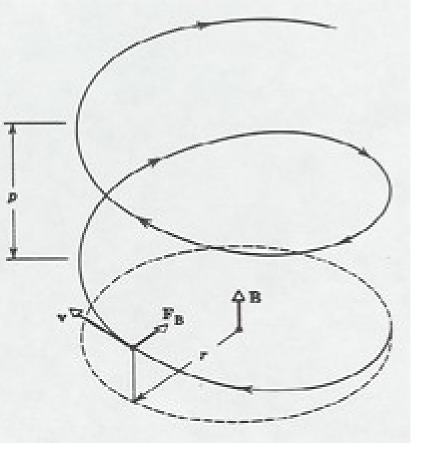
Synchrotron radiation – total emitted power

- *relativistic Larmor's formula*

$$\frac{dW}{dt}=\frac{2q^2}{3c^3}\gamma^4\left(a_\perp^2+\gamma^2 a_\parallel^2\right)$$

- frequency of rotation

$$\omega_B=\frac{qB}{\gamma m c}=\frac{1}{\gamma}\omega_L \quad , \quad a_\perp=\omega_B v_\perp$$



- emitted power (pitch angle θ)

$$\frac{dW}{dt}=\frac{2q^4}{3m^2c^5}\gamma^2 B^2\beta^2\sin^2\theta$$

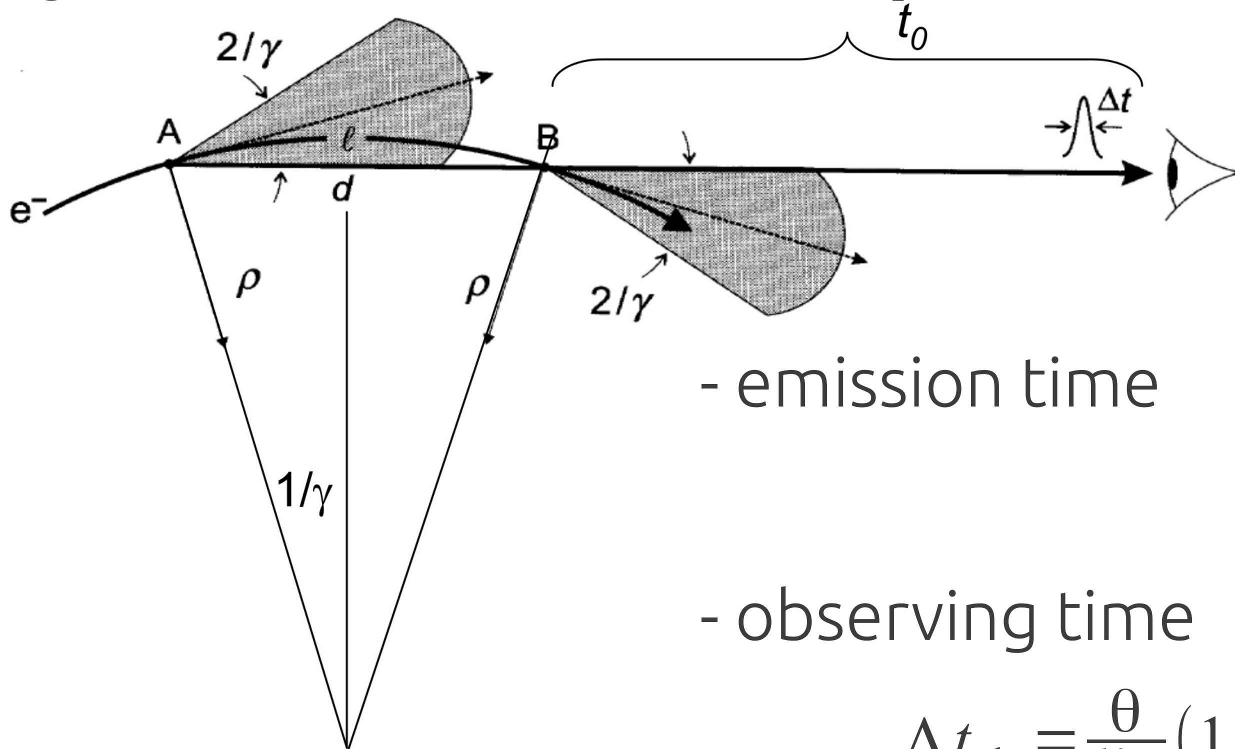
for isotropic distribution of electrons

$$\frac{dW}{dt}=\frac{4}{3}c\,\sigma_T u_B \beta^2\gamma^2$$

- cooling timescale

$$\tau=\frac{E}{dW/dt}\propto\frac{1}{B^2 E}$$

Synchrotron radiation – spectrum of a single electron



- emission time

$$\Delta t_e = \frac{\theta}{\omega_B}$$

- observing time

$$\Delta t_{\rm obs} = \frac{\theta}{\omega_{\rm B}}(1-\beta) \quad , \quad 1-\beta \approx \frac{1}{2\gamma^2}$$

$$\Delta t_{\rm obs} = \frac{1}{\gamma^3 \omega_{\rm B}} = \frac{1}{\gamma^2 \omega_{\rm L}}$$

- characteristic frequency

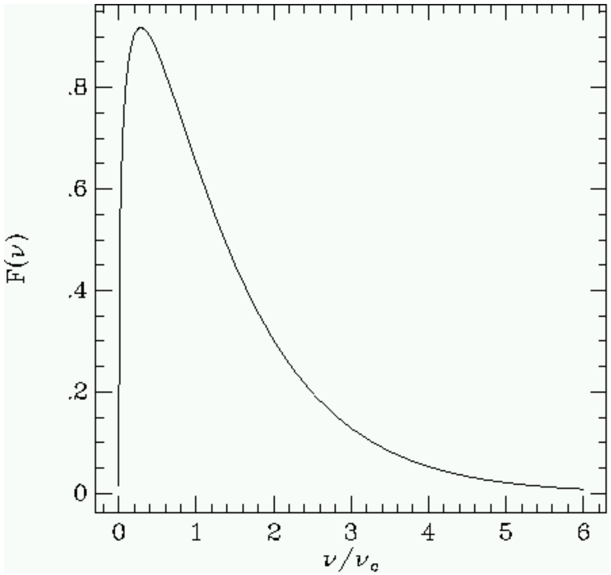
$$\omega_c \sim \frac{1}{\Delta t_{\rm obs}} = \gamma^3 \omega_{\rm B} = \gamma^2 \omega_{\rm L} = \gamma^2 \frac{q B}{m c}$$

Synchrotron radiation – power law distribution

- power law distribution of electrons

$$dN = N_0 \gamma^{-p} d\gamma$$

- spectrum of a single electron



$$\frac{dW_\omega}{dt}(\gamma) = \frac{4}{3} \sigma_T c u_B \gamma^2 F_\omega(\gamma)$$

- spectrum of electron distribution

$$F_\omega(\gamma) \approx \delta(\omega - \omega_c)$$

$$\frac{dW_\omega}{dt} = \int \frac{dW_\omega}{dt}(\gamma) dN$$

$$\frac{dW_\omega}{dt} = \frac{C}{2\omega_L} \left(\frac{\omega}{\omega_L}\right)^{-\frac{p-1}{2}}$$

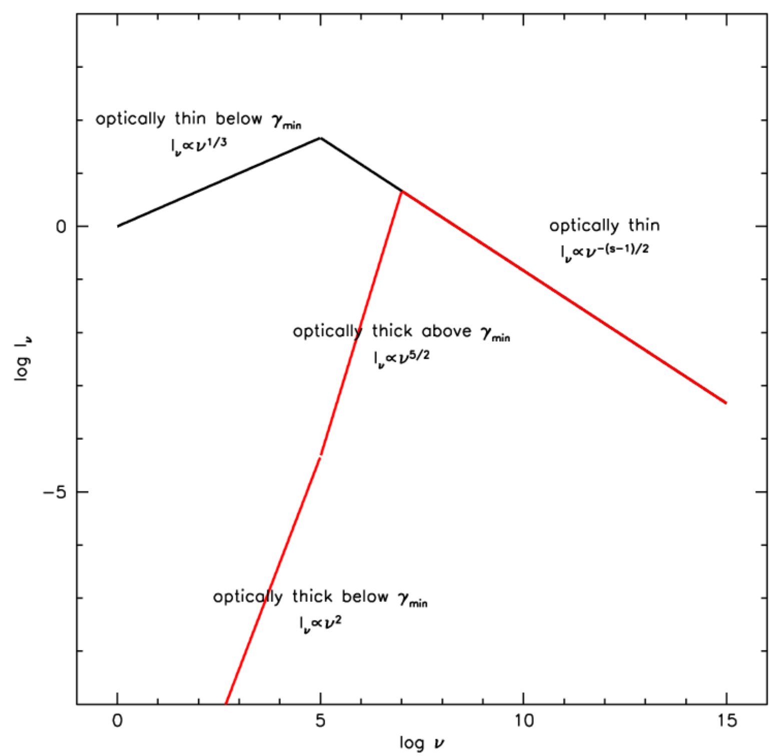
- spectral index (p-1)/2

Synchrotron radiation

- degree of polarization

$$\Pi = \frac{p+1}{p+7/3}$$

- synchrotron self-absorption

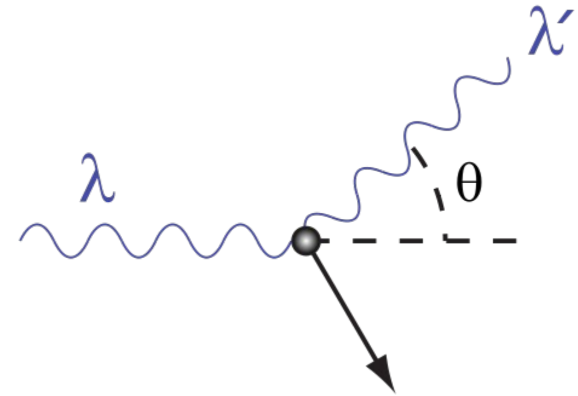


Compton scattering

- Thomson scattering

$$\epsilon = \epsilon_0 \quad , \quad \frac{d\sigma}{d\Omega} = \frac{1}{2} r_o^2 \left(1 + \cos^2\theta\right) \quad , \quad \sigma_T = \frac{8\pi}{3} r_0^2$$

- Compton scattering



$$\epsilon = \frac{\epsilon_0}{1 + \frac{\epsilon}{mc^2}\left(1 - \cos\theta\right)}$$

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2} \frac{\epsilon^2}{\epsilon_0^2} \left(\frac{\epsilon_0}{\epsilon} + \frac{\epsilon}{\epsilon_0} - \sin^2\theta \right)$$

$$\sigma = \sigma_T \begin{cases} 1 - 2\mathrm{x} + \ldots & \text{for } x \ll 1 \\ \frac{3}{8} x^{-1} \left(\ln 2\mathrm{x} + \frac{1}{2} \right) & \text{for } x \gg 1 \end{cases} \quad , \quad x \stackrel{\text{def}}{=} \frac{\epsilon}{mc^2}$$

Compton scattering from electron in motion

- lab frame vs. electron frame

$$\epsilon_0{}' = \epsilon_0 \gamma (1 - \beta \cos\theta)$$

$$\epsilon = \epsilon' \gamma (1 + \beta \cos\theta)$$

$$\epsilon' = \frac{\epsilon_0{}'}{1 + \frac{\epsilon_0{}'}{m c^2}(1 - \cos\Theta)} \approx \epsilon_0{}' \left[1 - \frac{\epsilon_0{}'}{m c^2}(1 - \cos\Theta) \right]$$

- energy transfer

$$\epsilon \approx \gamma \epsilon' \approx \gamma \epsilon_0{}' \approx \gamma^2 \epsilon_0$$

- maximal energy

$$\epsilon < \gamma m c^2 + \epsilon_0$$

Compton scattering from electron in motion

- single scattering power

$$\frac{dW}{dt} = \frac{4}{3} c \sigma_T u_{rad} \beta^2 \gamma^2$$

- similar to synchrotron

$$\frac{\dot{\gamma}_{synch}}{\dot{\gamma}_{Comp}} = \frac{u_B}{u_{rad}}$$

- cooling timescale

$$\tau_{Comp} = \frac{\gamma}{\dot{\gamma}} \propto \frac{1}{u_{rad} \gamma}$$

- spectral index for a power-law electron distribution

$$\alpha = \frac{p-1}{2}$$

Exercises

1. Derive the inverse Compton power for single scattering

$$\frac{dW}{dt} = \frac{4}{3} c \sigma_T u_{\rm rad} \beta^2 \gamma^2$$

(see e.g. Blumenthal & Gould 1970)