

Introduction to general relativity

Andrzej Krasiński

N. Copernicus Astronomical Centre, Polish Academy of Sciences,
Bartycka 18, 00 716 Warszawa, Poland, email: akr@camk.edu.pl

Notes to lectures based on the book:

Jerzy Plebański and Andrzej Krasiński

An introduction to general relativity and cosmology.

Cambridge University Press 2006, 534 pp.

Contents

1	How the theory of relativity came into being (a brief historical sketch.)	11
1.1	Special vs. general relativity.	11
1.2	Space and inertia in Newtonian physics.	11
1.3	Newton's theory and the orbits of planets.	12
1.4	The basic assumptions of general relativity.	13
I	ELEMENTS OF DIFFERENTIAL GEOMETRY	17
2	A short sketch of two-dimensional differential geometry.	19
2.1	Constructing parallel straight lines in a flat space.	19
2.2	Generalizing the notion of parallelism to curved surfaces.	20
3	Tensors, tensor densities.	23
3.1	What are tensors good for?	23
3.2	Differentiable manifolds.	23
3.3	Scalars.	25
3.4	Contravariant vectors.	25
3.5	Covariant vectors.	26
3.6	Tensors of second rank.	26
3.7	Tensor densities.	27
3.8	Tensor densities of arbitrary rank.	28
3.9	Algebraic properties of tensor densities.	28
3.10	The completely antisymmetric symbol.	29
3.11	Multidimensional Kronecker deltas.	31
3.12	Examples of applications of the Levi-Civita symbol and of the multidimensional Kronecker delta.	32

3.13 Exercises.	33
4 Covariant derivatives.	35
4.1 Differentiation of tensors.	35
4.2 Axioms of the covariant derivative.	37
4.3 A field of vector bases on a manifold and scalar components of tensors.	38
4.4 The affine connection.	39
4.5 The explicit formula for the covariant derivative of tensor densities.	40
4.6 Exercises.	41
5 Parallel transport and geodesic lines.	43
5.1 Parallel transport.	43
5.2 Geodesic lines.	44
5.3 Exercises.	45
6 Curvature of a manifold; flat manifolds.	47
6.1 The commutator of second covariant derivatives.	47
6.2 The relation between curvature and parallel transport.	49
6.3 Covariantly constant fields of vector bases.	54
6.4 A torsion-free flat manifold.	54
6.5 Parallel transport in a flat manifold.	55
6.6 Geodesic deviation.	55
6.7 Algebraic and differential identities obeyed by the curvature tensor.	56
6.8 Exercises.	57
7 Riemannian geometry.	59
7.1 The metric tensor.	59
7.2 Riemann spaces.	60
7.3 Signature of a metric, degenerate metrics.	60
7.4 Christoffel symbols.	61
7.5 Curvature of a Riemann space.	62
7.6 Flat Riemann spaces.	63
7.7 Subspaces of a Riemann space.	63
7.8 The Riemann curvature vs. the normal curvature of a surface.	64

CONTENTS	5
----------	---

7.9	The geodesic line as the line of extremal distance.	64
7.10	Timelike, null and spacelike intervals in a 4-dimensional spacetime.	65
7.11	Embedding a Riemann space V_n in a Riemann space V_{n+1}	67
7.12	Exercises	73
8	Symmetries of Riemann spaces, invariance of tensors.	75
8.1	Symmetry transformations.	75
8.2	The Killing equations.	75
8.3	The connection between generators and the invariance transformations. . .	78
8.4	Finding the Killing vector fields.	78
8.5	Invariance of other tensor fields.	80
8.6	The Lie derivative.	80
8.7	The algebra of Killing vector fields.	82
8.8	Spherically symmetric 4-dimensional Riemann spaces.	82
8.9	Exercises	85
9	The spatially homogeneous Bianchi-type spacetimes.	87
9.1	The Bianchi classification of 3-dimensional Lie algebras.	87
9.2	The dimension of the group vs. the dimension of the orbit.	92
9.3	Action of a group on a manifold.	93
9.4	Groups acting transitively, homogeneous spaces.	94
9.5	Invariant vector fields.	94
9.6	The metrics of the Bianchi-type spacetimes.	96
9.7	The isotropic Bianchi type spacetimes.	97
9.8	The Robertson–Walker metric – a formal derivation and full groups of symmetries.	98
9.9	Exercises	101
II	THE GRAVITATION THEORY	103
10	The Einstein equations and the sources of a gravitational field.	105
10.1	Why Riemannian geometry?	105
10.2	Local inertial frames.	105
10.3	Trajectories of free motion in Einstein’s theory.	106

10.4 Special relativity vs. gravitation theory.	109
10.5 The Newtonian limit of relativity.	109
10.6 Sources of the gravitational field.	110
10.7 The Einstein equations.	110
10.8 Hilbert's derivation of the Einstein equations.	112
10.9 The Newtonian limit of Einstein's equations.	113
10.10 Examples of sources in the Einstein equations: perfect fluid and dust.	117
10.11 Equations of motion of a perfect fluid.	119
10.12 The cosmological constant.	119
10.13 Matching solutions of Einstein's equations	121
10.14 Exercises.	123
11 Spherically symmetric gravitational field of isolated objects.	125
11.1 The curvature coordinates.	125
11.2 The Schwarzschild solution	128
11.3 Orbits of planets in the gravitational field of the Sun.	131
11.4 Deflection of light-rays in the Schwarzschild field.	137
11.5 Measuring the deflection of light-rays.	140
11.6 Gravitational lenses.	143
11.7 The spurious singularity of the Schwarzschild solution at $r = 2m$	145
11.8 Interpretation of the spurious singularity at $r = 2m$; black holes.	149
11.9 The Schwarzschild solution in other coordinate systems.	152
11.10 The equation of hydrostatic equilibrium	154
11.11 The 'interior Schwarzschild solution'	156
11.12 Exercises.	157
12 Relativistic hydrodynamics	159
12.1 Motion of a continuous medium in Newtonian mechanics	159
12.2 Motion of a continuous medium in relativistic mechanics.	161
12.3 The equations of evolution of θ , $\sigma_{\alpha\beta}$, $\omega_{\alpha\beta}$ and u^α ; the Raychaudhuri equation.	164
12.4 Singularities and singularity theorems	167
12.5 Exercises	168
13 Relativistic cosmology I: general geometry.	169

13.1 A continuous medium as a model of the Universe.	169
13.2 Optical observations in the Universe – part I	170
13.2.1 The geometric optics approximation	170
13.2.2 The redshift	172
13.3 The optical tensors.	174
13.4 The apparent horizon.	176
13.5 Optical observations in the Universe – part II	176
13.5.1 The area distance	176
13.5.2 The reciprocity theorem	178
13.6 Exercises.	182
14 Relativistic cosmology II: the Robertson–Walker geometry.	183
14.1 The Robertson–Walker metrics as models of the Universe.	183
14.2 Optical observations in an R–W Universe	185
14.2.1 The redshift	185
14.2.2 The redshift–distance relation	186
14.3 The Friedmann equations	187
14.4 The Λ CDM model	188
14.5 The redshift drift – a test for accelerated expansion	189
14.6 The Friedmann solutions with $\Lambda = 0$	190
14.6.1 The redshift – distance relation in the $\Lambda = 0$ Friedmann models . .	191
14.7 The Newtonian cosmology.	192
14.8 The Friedmann solutions with the cosmological constant.	194
14.9 Horizons in the Robertson–Walker models.	196
14.10 The ‘history of the Universe’	203
14.11 The redshift – distance relation in the $\Lambda \neq 0$ Friedmann models	205
14.12 Exercises.	206
15 Relativistic cosmology III: the Lemaître–Tolman geometry.	207
15.1 The comoving-synchronous coordinates.	207
15.2 Spherically symmetric inhomogeneous models.	208
15.3 The Lemaître–Tolman model.	209
15.4 Conditions of regularity at the centre.	212

15.5 Formation of voids in the Universe.	213
15.6 Formation of other structures in the Universe.	214
15.7 The influence of cosmic expansion on planetary orbits	215
15.8 Apparent horizons in the L–T model.	217
15.9 The redshift.	219
15.10 The blueshift	219
15.11 Properties of singularities	221
15.12 Increasing and decreasing density perturbations.	222
15.13 Mimicking the accelerating expansion of the Universe by an inhomogeneous mass distribution	222
15.14 Drift of light rays	225
15.15 Exercises.	226
16 Relativistic cosmology IV: The Szekeres models	229
17 The Kerr solution.	231
17.1 The Kerr metric in the original form.	231
17.2 Basic properties.	232
17.3 The event horizons and the stationary limit hypersurfaces.	237
17.4 The Hamiltonian and the Poisson bracket	240
17.5 General geodesics.	242
17.6 Geodesics in the equatorial plane.	246
17.7 The Penrose process.	253
17.8 Stationary-axisymmetric spacetimes and locally non-rotating observers. . .	255
17.9 A source of the Kerr field?	258
17.10 Exercises	258
18 Relativity enters technology – the Global Positioning System	261
18.1 Purpose and setup	261
18.2 The principle of position determination	262
18.3 The reference frames and the Sagnac effect	263
18.4 Earth’s gravitation and the SI time units	265
18.5 The realisation of the coordinate time	266
18.6 Selected corrections of the orbits of the GPS satellites	267

CONTENTS	9
18.6.1 Corrections for gravity and velocity	267
18.6.2 The eccentricity correction	270
18.7 The 9 largest relativistic effects in the GPS	271
19 Subjects omitted in this course	273